AGM LECTURE NOTES

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Contents

1. AGM in Real Numbers

We will first define arithmetic-geometric mean(agm) of two numbers positive real numbers a and b. This is the common limit of two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, where sequences are defined by:

$$\begin{aligned} &a_0 = a, \quad b_0 = b \\ &a_{n+1} = (a_n + b_n)/2 \\ &b_{n+1} = (a_n b_n)^{1/2} \end{aligned} \tag{1.1}$$

First observation is $a_n \geq b_n$ for all n.

$$a_{n+1} \geq b_{n+1} \iff \frac{a_n + b_n}{2} \geq \sqrt{a_n b_n} \iff a_n^2 + b_n^2 - 2a_n b_n \geq 0 \tag{1.2}$$

Because $(a_n - b_n)^2 \geq 0$ is true always, $a_{n+1} \geq b_{n+1}$ is true for all $n > 0$. Without any loss of generality let $a \geq b$ so that $a_n \geq b_n$ is true for all n.

Not only $a_n \geq b_n$ is true but;

$$\begin{aligned} a_{n+1} \leq a_n \iff \frac{a_n + b_n}{2} \leq a_n \iff b_n \leq a_n \\ b_n \leq b_{n+1} \iff b_n \leq \sqrt{a_n b_n} \iff b_n \leq a_n \end{aligned} \tag{1.3}$$

From (1.3) we obtain;

$$b_n \leq b_{n+1} \leq a_{n+1} \leq a_n \tag{1.4}$$

Now it is left to show that they converge to common limit.

$$0 \leq a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{1}{2}\left(\sqrt{a_n} - \sqrt{b_n}\right)^2 = \frac{1}{2}\frac{(a_n - b_n)^2}{(\sqrt{a_n} + \sqrt{b_n})^2} \tag{1.5}$$

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This shows that

$$0 \leq a_{n+1} - b_{n+1} \leq \frac{1}{2}(a_n - b_n) \leq 2^{-n}(a - b) \tag{1.6}$$

Hence $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exists and equal. We will denote this common limit as;

$$M(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \tag{1.7}$$

There are two obvious but useful properties of agm:

$$\begin{aligned} &M(a,b) = M(a_1,b_1) = M(a_2,b_2) = \dots \\ &M(\lambda a, \lambda b) = \lambda M(a,b) \end{aligned} \tag{1.8}$$

Note 1. It is useful to note a quantity $c_{n+1} := \frac{1}{2}(a_n - b_n)$. From this definition we can show;

$$4a_n c_n = (a_{n-1} + b_{n-1})(a_{n-1} - b_{n-1}) = \frac{(a_{n-2} - b_{n-2})^2}{4} = c_{n-1}^2 \tag{1.9}$$

Or in convenient form;

$$c_n = \frac{c_{n-1}^2}{4a_n} \tag{1.10}$$

We now give a definition for pth-order convergence and show that agm converges quadratically.

Definition 1. We say $\alpha_n \to \alpha$ with pth-order convergence if

$$\left| \frac{\alpha_{n+1} - \alpha}{(\alpha_n - \alpha)^p} \right| = \mathcal{O}(1) \tag{1.11}$$

where $\mathcal{O}(1)$ is usual big-O notation. Moreover if α_n are functions defined for all x in a set K, and constant is independent of x then we say that convergence is *uniformly pth order*.

Roughly speaking, quadratic convergence doubles the number of digits that agree between successive iterates and the limit.

Let $\alpha_n = a_n - b_n$ and hence $\alpha_n \to 0$.

$$\begin{aligned} \left| \frac{\alpha_{n+1}}{\alpha_n^2} \right| &= \left| \frac{1}{2} \frac{(a_n - b_n)^2}{(\sqrt{a_n} + \sqrt{b_n})^2} \frac{1}{(a_n - b_n)^2} \right| \\ &= \frac{1}{2} \left| \frac{1}{(\sqrt{a_n} + \sqrt{b_n})^2} \right| \leq \frac{1}{2b} \end{aligned} \tag{1.12}$$

From (1.12), a and b restricted to compact subsets of $(0, \infty)$ implies uniform quadratic convergence for $M(a,b)$.

Now we state and prove the theorem that relates agm to elliptic integrals and then we will work on properties.

Theorem 1. *If $a \geq b > 0$, then*

$$M(a,b) \int_0^{\pi/2} (a^2 \cos^2 \phi + b^2 \sin^2 \phi)^{-1/2} d\phi = \frac{\pi}{2} \tag{1.13}$$

3 different proof are given for theorem (1) in seperate subsections. Rest of the discussion will refer to first proof.

An application of theorem (1) is the arc length of the lemniscate $r^2 = \cos 2\theta$:

$$4 \int_0^{\pi/4} (r^2 + (dr/d\theta)^2)^{1/2} d\theta = 4 \int_0^{\pi/4} (\cos 2\theta)^{-1/2} d\theta \tag{1.14}$$

Substitute $\cos 2\theta = cos^2\phi$;

$$4 \int_0^{\pi/2} (1 + \cos^2 \phi)^{-1/2} d\phi = 4 \int_0^{\pi/2} (2\cos^2 \phi + \sin^2 \phi)^{-1/2} d\phi = M(\sqrt{2}, 1) \tag{1.15}$$

Our observations can be related to classical theory of complete elliptic integrals of the first kind, i.e. integrals of the form;

$$K(k,\pi/2)=\int_0^{\pi/2}(1-k^2\sin^2\phi)^{-1/2}d\phi=\int_0^1((1-z^2)(1-k^2z^2))^{-1/2}dz \tag{1.16}$$

Substitution $z=\sin\phi$, $dz=\cos\phi d\phi=\sqrt{1-z^2}d\phi$ shows the second equality. Now let $k=\frac{a-b}{a+b}$;

$$\frac{2\sqrt{k}}{1+k}=2\sqrt{\frac{a-b}{a+b}}\frac{a+b}{2a}=\frac{\sqrt{a^2-b^2}}{a}\quad\text{and}\quad k=\frac{\sqrt{a_1^2-b_1^2}}{a_1} \tag{1.17}$$

Observe $I(a,b)=a^{-1}K(\frac{2\sqrt{k}}{1+k},\pi/2)$:

$$\begin{aligned}a^{-1}K\left(\frac{2\sqrt{k}}{1+k},\pi/2\right)&=a^{-1}\int_0^{\pi/2}a(a^2-(a^2-b^2)\sin^2\theta)\\&=\int_0^{\pi/2}(a^2\cos^2\phi+b^2\sin^2\phi)d\phi\end{aligned} \tag{1.18}$$

and $I(a_1,b_1)=a_1^{-1}K(k,\pi/2)$ follows exactly from (1.17). Now (1.48) is equivalent to;

$$K\left(\frac{2\sqrt{k}}{1+k},\pi/2\right)=(1+k)K(k,\pi/2) \tag{1.19}$$

Using (1.22) will give the well known agm relation;

$$f(x)=\frac{1+x}{2}f\left(\frac{2\sqrt{x}}{1+x}\right) \tag{1.20}$$

Note that substitution (1.50) can be written as;

$$\sin\phi=\frac{(1+k)\sin\theta}{1+k\sin^2\theta}=\frac{2a\sin\theta}{a+b+(a-b)\sin^2\theta} \tag{1.21}$$

which is now called Gauss Transformation.

We can restate Theorem 1 for $0\le k<1$ as;

$$\frac{1}{M(1+k,1-k)}=\frac{2}{\pi}\int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}d\theta=\frac{2}{\pi}K(k,\pi/2) \tag{1.22}$$

Proof.

$$\begin{aligned}M(1+k,1-k)^{-1}&=M(1,\sqrt{1-k^2})^{-1}\\&=\frac{2}{\pi}\int_0^{\pi/2}(\cos^2\phi+(1-k^2)\sin^2\phi)^{-1/2}d\phi\\&=\frac{2}{\pi}\int_0^{\pi/2}(1-k^2\sin^2\phi)^{-1/2}d\phi\end{aligned} \tag{1.23}$$

□

Finally set $k'=\sqrt{1-k^2}$ and rewrite (1.22) as;

$$\frac{1}{M(1,k')}=\frac{2}{\pi}\int_0^{\pi/2}(1-k^2\sin^2\phi)^{-1/2}d\phi \tag{1.24}$$

Gauss‘s interpretation is; average value of the function $(1-k^2\sin^2\phi)^{-1/2}$ on the interval $[0,\pi/2]$ is the reciprocal of the agm of the reciprocals of the minimum and maximum values of the function.

We now have an integral represantation of agm, but series representation is know for integral in (1.24) and given by;

$$\frac{1}{M(1,k')}=\sum_{n=0}^{\infty}\left(\frac{1\cdot3\cdot5\cdots(2n-1)}{n!2^n}\right)^2k^{2n} \tag{1.25}$$

Proof. First of all it is necessary to find;

$$\begin{aligned}\int_0^{\pi/2}(\sin\phi)^{2n}d\phi &= \frac{1}{(2i)^{2n}}\int_0^{\pi/2}(e^{i\phi}-e^{-i\phi})^{2n}d\phi \\ &= \frac{1}{(2i)^{2n}}\int_0^{\pi/2}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}e^{i\phi(2n-k)}e^{-i\phi k}d\phi \\ &= \frac{1}{(2i)^{2n}}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\int_0^{\pi/2}e^{2i\phi(n-k)}d\phi\end{aligned} \tag{1.26}$$

Now it is important to notice that integral of exponential will be different if $n = k$.[1]

$$\int_0^{\pi/2}e^{2i\phi(n-k)}d\phi = \begin{cases}\frac{(-1)^{n-k}}{2i(n-k)} - \frac{1}{2i(n-k)} & \text{if } n \neq k \\ \frac{\pi}{2} & \text{if } n = k\end{cases} \tag{1.27}$$

Key observation is that under the summation over k, terms that $k \neq n$ will cancel each other. Hence only $n = k$ term left. And also $(-1)^n$ terms cancels out. Then we get;

$$\int_0^{\pi/2}(\sin\phi)^{2n}d\phi = \frac{1}{2^{2n}}\frac{\pi}{2}\binom{2n}{n} = \frac{1}{2^{2n}}\frac{\pi}{2}\frac{(2n)!}{(n!)^2} = \frac{\pi}{2}\frac{1}{2^n n!}\prod_{k=1}^{n}(2k-1) \tag{1.28}$$

We also need to show one direct identity.

$$\begin{aligned}\frac{1}{2^n}\frac{(2n)!}{n!} = \prod_{k=1}^{n}(2k-1) &= \prod_{k=1}^{2n}k\prod_{k=1}^{n}(2k)^{-1} \\ &= \frac{(2n)!}{2^n n!}\end{aligned} \tag{1.29}$$

Now lets turn back to integral representation of agm. Recall that $0 \leq k < 1$ hence we can expand root in taylor series and it is uniformly convergent.

$$\begin{aligned}&\frac{2}{\pi}\int_0^{\pi/2}(1-k^2\sin^2\phi)^{-1/2}d\phi \\ &\qquad = \frac{2}{\pi}\int_0^{\pi/2}\sum_{n=0}^{\infty}\left(\frac{1\cdot 2\cdot 3\cdots(2n-1)}{n!2^n}\right)k^{2n}(\sin\phi)^{2n} \\ &\qquad = \frac{2}{\pi}\sum_{n=0}^{\infty}\left(\frac{1\cdot 2\cdot 3\cdots(2n-1)}{n!2^n}\right)k^{2n}\int_0^{\pi/2}(\sin\phi)^{2n}\end{aligned} \tag{1.30}$$

Use (1.28) and proof is done for series representation of agm. □

We can observe from series representation of agm that it solves hypergeometric differential equation;

$$(k^3-k)y'' + (3k^2-1)y' + ky = 0 \tag{1.31}$$

Proof. Lets first find derivatives of series representation;

$$\begin{aligned}y' &= \sum_{n=0}^{\infty}2n\left(\frac{1\cdot 2\cdots(2n-1)}{n!2^n}\right)^2k^{2n-1} \\ y'' &= \sum_{n=0}^{\infty}2n(2n-1)\left(\frac{1\cdot 2\cdots(2n-1)}{n!2^n}\right)^2k^{2n-2}\end{aligned} \tag{1.32}$$

[1]To make it clear, here k is a summation variable and not related to k in agm.

(1.31) turns into:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{1\cdot 2\cdots(2n-1)}{n!2^n}\right)^2\left[(2n+1)^2k^{2n+1}-4n^2k^{2n-1})\right]=0\\
&\sum_{n=0}^{\infty}\left(\frac{1\cdot 2\cdots(2n-1)}{n!2^n}\right)^2\left[(2n+1)^2k^{2n+1}\right]\\
&\quad=\sum_{n=0}^{\infty}\left(\frac{1\cdot 2\cdots(2n-1)}{n!2^n}\right)^2\left[4n^2k^{2n-1}\right]\\
&\qquad=\sum_{n=0}^{\infty}\left(\frac{1\cdot 2\cdots(2n-1)}{n!2^n}\right)^2\left[(2n+1)^2k^{2n+1}\right]
\end{aligned}\tag{1.33}$$

This proves that $y(k)=(M(1+k,1-k))^{-1}=(M(1,\sqrt{1-k^2}))^{-1}$ solves the equation. Moreover we can show that $(M(1,k))^{-1}$ also a solution; Let $b=\sqrt{1-k^2}$. Then observe $k=\sqrt{1-b^2}$. Hence;

$$y\left(\sqrt{1-k^2}\right)=\frac{1}{M(1,b)}=y(b)\tag{1.34}$$

It is sufficient to show that $y(\sqrt{1-k^2})$ is also a solution to (1.31).

$$\begin{aligned}
\frac{\partial}{\partial k}y(b)&=\frac{\partial}{\partial b}y(b)\frac{\partial}{\partial k}b=y'(b)(-k)(1-k^2)^{-1/2}=-y'\sqrt{1-b^2}/b\\
\frac{\partial^2}{\partial k^2}y(b)&=\frac{\partial^2}{\partial b^2}y(b)\left(\frac{\partial}{\partial k}b\right)^2+\frac{\partial}{\partial b}y(b)\frac{\partial^2}{\partial k^2}b\\
&=y''(b)k^2(1-k^2)^{-1}-y'(b)\left((1-k^2)^{-1/2}+k^2(1-k^2)^{-3/2})\right)\\
&=y''(b)(1-b^2)/b^2-y'(b)\left(1/b+(1-b^2)/b^3\right)
\end{aligned}\tag{1.35}$$

Transform (1.31) to variable b and scale equation by $b/\sqrt{1-b^2}$;

$$b(b^2-1)y''(b)+y'(b)-y'(b)(2-3b^2)+by(b)=0\tag{1.36}$$

Therefore $(M(1,b))^{-1}$ also solves the equation. General solution to (1.31) is given by;

$$y(k)=\frac{A}{M(1+k,1-k)}+\frac{B}{M(1,k)}\tag{1.37}$$

□

It is also important to see how Gauss found this differential equation. In his works; from series representations he note quantities y, ky' and $k^2y''+ky'$. Then it was obvious for him to observe differential equation. We can observe last quantity‘s summation variable can be shifted, and then it is easy to observe;

$$k^2y''(k)+3ky'(k)+y(k)=\frac{1}{k^2}\left(k^2y''(k)+ky'(k)\right)\tag{1.38}$$

which is equivalent to (1.31).

We analysed complete elliptic integrals of the first kind and its relation to AGM. We will introduce complete eliptic integrals of the second kind and observe some relations with first kind.

$$E(k,\pi/2)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta=\int_0^1\frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\tag{1.39}$$

Finding series representation of $E(k,\pi/2)$ is very similar to $K(k,\pi/2)$. It is required to expand $(1-x)^{1/2}$ and then end result is given as;

$$E(k,\pi/2)=\frac{\pi}{2}\left\{1-\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{n!2^n}\right)^2\frac{k^{2n}}{2n-1}\right\}\tag{1.40}$$

Integrals of the first kind and the second are related by;

$$\begin{aligned}\frac{dE}{dk} &= \frac{E-K}{k}\\ \frac{dK}{dk} &= \frac{E-(1-k^2)K}{k(1-k^2)}\end{aligned} \tag{1.41}$$

Proof. For first relation, differentiate (1.39);

$$\frac{dE}{dk} = \int_0^{\pi/2} \frac{-k\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta \tag{1.42}$$

and right hand side is;

$$\begin{aligned}\frac{E-K}{k} &= \frac{1}{k}\int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta} - \frac{1}{\sqrt{1-k^2\sin^2\theta}}\\ &= \int_0^{\pi/2} \frac{-k\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}}\end{aligned} \tag{1.43}$$

For the second relation, let us use series representation of K and E. Starting again from left hand side;

$$\begin{aligned}(k-k^3)\frac{dK}{dk} &= \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{n!2^n}\right)^2 (2n)k^{2n}\\ &\qquad - \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{n!2^n}\right)^2 (2n)k^{2n+2}\\ &= \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{n!2^n}\right)^2 \frac{k^{2n+2}}{2(n+1)}\end{aligned} \tag{1.44}$$

And;

$$\begin{aligned}E-(1-k^2)K &= -\frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{n!2^n}\right)^2 \left(\frac{2n}{2n-1}\right)k^{2n}\\ &\qquad + \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{n!2^n}\right)^2 k^{2n+2}\end{aligned} \tag{1.45}$$

Shift the summation variable of the upper sumation;

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{n!2^n}\right)^2 \left(\frac{2n}{2n-1}\right)k^{2n}\\ &\qquad = \sum_{n=0}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{n!2^n}\right)^2 \frac{(2n+1)}{2(n+1)}k^{2n+2}\end{aligned} \tag{1.46}$$

Hence;

$$E-(1-k^2)K = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{n!2^n}\right)^2 \frac{k^{2n+2}}{2(n+1)} \tag{1.47}$$

□

1.1. **First proof of Theorem 1.**

Proof. Let $I(a,b)$ represent integral expression. It is sufficient to show that

$$I(a,b) = I(a_1,b_1) \tag{1.48}$$

which implies $I(a,b) = \lim_{n\to\infty} I(a_n,b_n) = I(M(a,b),M(a,b))$. Then observe,

$$I(M(a,b),M(a,b)) = M(a,b)^{-1}\int_0^{\pi/2} d\phi = \frac{\pi}{2}M(a,b)^{-1} \tag{1.49}$$

To show (1.48), we need to make a change of variable. This is introduced by Gauss.

$$\sin\phi = \frac{2a\sin\theta}{a+b+(a-b)\sin^2\theta} \tag{1.50}$$

To be sure that this expression is well defined,

$$\frac{2a\sin\theta}{a+b+(a-b)\sin^2\theta} \le \frac{2\sin\theta}{(1+\sin^2\theta)} \le 1 \tag{1.51}$$

and attains 0 and 1 at $\theta = 0, \pi/2$. Moreover, one should also check $x = \sin\theta$ derivative to be sure that function is always increasing.

We should show that;

$$\cos\phi = \frac{2\cos\theta(a_1^2\cos^2\theta + b_1^2\sin^2\theta)^{1/2}}{a+b+(a-b)\sin^2\theta} \tag{1.52}$$

Proof. We will use the identity $\cos^2\phi + \sin^2\phi = 1$, but first

$$a_1^2 - b_1^2 = \left(\frac{a+b}{2}\right)^2 - ab = \frac{(a-b)^2}{4} \tag{1.53}$$

Now writing the identity by using (1.50) and (1.52) results;

$$(a+b+(a-b)\sin^2\theta)^2 - (2a\sin\theta)^2 = 4\cos^2\theta(a_1^2\cos^2\theta + b_1^2\sin^2\theta) \tag{1.54}$$

Manipulate terms side by side:

$$\begin{aligned} &((a+b)+(a-b)\sin^2\theta)^2 - 4a^2\sin^2\theta \\ &= (a+b)^2 + 2(a^2-b^2-2a^2)\sin^2\theta + (a-b)^2\sin^4\theta \\ &= (a+b)^2 - 2(a^2+b^2)\sin^2\theta + (a-b)^2\sin^4\theta \end{aligned} \tag{1.55}$$

and;

$$\begin{aligned} &4\cos^2\theta(a_1^2\cos^2\theta + b_1^2\sin^2\theta) \\ &= 4\cos^2\theta(a_1^2 - (a_1^2-b_1^2)\sin^2\theta) \\ &= \cos^2\theta((a+b)^2 - (a-b)^2\sin^2\theta) \\ &= (a+b)^2 - (a-b)^2\sin^2\theta - (a+b)^2\sin^2\theta + (a-b)^2\sin^4\theta \\ &= (a+b)^2 - 2(a^2+b^2)\sin^2\theta + (a-b)^2\sin^4\theta \end{aligned} \tag{1.56}$$

□

Next we need to show;

$$(a^2\cos^2\phi + b^2\sin^2\phi)^{1/2} = a\frac{a+b-(a-b)\sin^2\theta}{a+b+(a-b)\sin^2\theta} \tag{1.57}$$

Proof. Starting from left-hand side;

$$(a^2\cos^2\phi + b^2\sin^2\phi)^{1/2} = (a^2 - (a^2-b^2)\sin^2\phi)^{1/2} \tag{1.58}$$

Use (1.50) and ignore denominator.

$$\begin{aligned} &= a((a+b+(a-b)\sin^2\theta)^2 - 4(a^2-b^2)\sin^2\theta)^{1/2} \\ &= a(a+b-(a-b)\sin^2\theta) \end{aligned} \tag{1.59}$$

□

Now differentiate (1.50);

$$\cos\phi d\phi = \frac{2a\cos\theta(a+b+(a-b)\sin^2\theta) - 4a(a-b)\cos\theta\sin^2\theta}{(a+b+(a-b)\sin^2\theta)^2}d\theta \tag{1.60}$$

Use (1.52) and manipulate terms;

$$
\begin{aligned}
(a_1^2\cos^2\theta + b_1^2\sin^2\theta)^{1/2}d\phi &= \frac{a(a+b+(a-b)\sin^2\theta) - 2a(a-b)\sin^2\theta}{a+b+(a-b)\sin^2\theta}d\theta \\
(a_1^2\cos^2\theta + b_1^2\sin^2\theta)^{1/2}d\phi &= a\frac{a+b-(a-b)\sin^2\theta}{a+b+(a-b)\sin^2\theta}d\theta \\
(a_1^2\cos^2\phi + b_1^2\sin^2\phi)^{-1/2}d\phi &= (a^2\cos^2\theta + b^2\sin^2\theta)^{-1/2}d\theta
\end{aligned}
\tag{1.61}
$$

This proves that integral does not change when a and b is iterated by agm. □

1.2. **Second proof of Theorem 1.** There is another proof of (1) which again uses change of variable, but in a different way and terms should has to be manipulated carefully.

Proof. Start with very similar integral;

$$
T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \tag{1.62}
$$

Then make a substitution $t = b\tan\theta$;

$$
\begin{aligned}
T(a,b) &= \frac{2}{\pi}\int_0^{\infty} \left(a^2\frac{b^2}{t^2+b^2} + b^2\frac{t^2}{t^2+b^2}\right)^{-1/2}\left(\frac{b}{t^2+b^2}\right)dt \\
&= \frac{2}{\pi}\int_0^{\infty} \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}}
\end{aligned}
\tag{1.63}
$$

Now substitute $u = \frac{1}{2}(t - ab/t)$ but first, terms should be arranged nicely;

$$
\begin{aligned}
&= \frac{2}{\pi}\int_0^{\infty} \frac{dt}{t\sqrt{(t - \frac{ab}{t} + \frac{a^2}{t} + \frac{ab}{t})(t - \frac{ab}{t} + \frac{b^2}{t} + \frac{ab}{t})}} \\
&= \frac{2}{\pi}\int_0^{\infty} \frac{dt}{2t\sqrt{(u + \frac{a(a+b)}{2t})(u + \frac{b(a+b)}{2t})}}
\end{aligned}
\tag{1.64}
$$

Now lets find the differential element:

$$
du = \frac{1}{2}\left(1 + \frac{ab}{t^2}\right)dt \tag{1.65}
$$

Hence;

$$
\begin{aligned}
2du\left(t + \frac{ab}{t}\right)^{-1} &= \frac{dt}{t} \\
2du\left(2u + \frac{2ab}{t}\right)^{-1} &= \frac{dt}{t} \\
\frac{du}{2}\left(u + \frac{ab}{t}\right)^{-1} &= \frac{dt}{2t}
\end{aligned}
\tag{1.66}
$$

Observe that limits for u is from $-\infty$ to ∞ and rewrite (1.64);

$$
\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{du}{(u + \frac{ab}{t})\sqrt{(u + \frac{a(a+b)}{2t})(u + \frac{b(a+b)}{2t})}} \tag{1.67}
$$

Lets first observe;

$$
\begin{aligned}
\left(u + \frac{a(a+b)}{2t}\right)\left(u + \frac{b(a+b)}{2t}\right) &= u^2 + \frac{u}{2t}(a+b)^2 + \frac{ab(a+b)^2}{4t^2} \\
&= u^2 + (a+b)^2/4
\end{aligned}
\tag{1.68}
$$

And also;

$$\begin{aligned}\left(u+\frac{ab}{t}\right)^2 &= u^2+\frac{u}{t}2ab+\frac{(ab)^2}{t^2}\\ &= u^2+ab-\frac{(ab)^2}{t^2}+\frac{(ab)^2}{t^2}\\ &= u^2+ab\end{aligned} \tag{1.69}$$

At the end, we get;

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{du}{\sqrt{(u^2+a_1^2)(u^2+b_1^2)}} \tag{1.70}$$

This proves $T(a,b)=T(a_1,b_1)$. □

1.3. **Third Proof of Theorem 1.** In this section, I will follow Gauss‘s work on ’Werke’, III, pg.366-369. We start with assuming that $(M(1+x,1-x))^{-1}$ is analytic in some neighbourhood of zero. Then it follows;

$$\frac{1}{M(1+x,1-x)}=d_0+d_1x^2+d_2x^4+d_3x^6+\cdots \tag{1.71}$$

where $d_0=1$ but I prefer to leave it as d_0. We also know;

$$\begin{aligned}\frac{1}{M(1+2\sqrt{x}/(1+x),1-2\sqrt{x}/(1+x)} &= \frac{1}{M(1,\sqrt{1-4x/(1+x)^2})}\\ &= \frac{1+x}{M(1+x,1-x)}\end{aligned} \tag{1.72}$$

This equality under series expansion can be written as;

$$(1+x)(d_0+d_1x^2+d_2x^4+\cdots)=d_0+d_1\left(\frac{2\sqrt{x}}{1+x}\right)^2+d_2\left(\frac{2\sqrt{x}}{1+x}\right)^4+\cdots \tag{1.73}$$

Let $x=t^2$ and multiply by $2t/(1+t^2)$;

$$2t(d_0+d_1t^4+d_2t^8+\cdots)=d_0\left(\frac{2t}{1+t^2}\right)+d_1\left(\frac{2t}{1+t^2}\right)^3+d_2\left(\frac{2t}{1+t^2}\right)^5+\cdots \tag{1.74}$$

Define $f_n(x)=(1+x)^{-n}$ and differentiate to find series expansion;

$$\begin{aligned}&f_n'(x)=-n(1+x)^{-n-1},\quad f_n''(x)=n(n+1)(1+x)^{-n-2},\\ &f_n'''(x)=-n(n+1)(n+2)(1+x)^{-n-3}\quad etc.\end{aligned} \tag{1.75}$$

Hence $f_n(x)$ in series is;

$$\begin{aligned}f_n(x) &= 1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3\cdots\\ &= \sum_{j=0}^{\infty}(-1)^j\frac{n(n+1)\cdots(n+j-1)}{j!}x^j\end{aligned} \tag{1.76}$$

On the right hand side of (1.74) each term can be represented as;

$$d_n(2t)^{2n+1}f_{2n+1}(t^2)=d_n2^{2n+1}\sum_{j=0}^{\infty}(-1)^j\frac{(2n+1)(2n+2)\cdots(2n+j)}{j!}t^{2n+2j+1} \tag{1.77}$$

Rewrite (1.74) as;

$$\begin{aligned}&d_0t+d_1t^5+d_2t^9+d_3t^{13}+\cdots=\\ &\sum_{n=0}^{\infty}d_n\frac{2^{2n}}{(2n)!}\sum_{j=0}^{\infty}(-1)^j\frac{(2n+j)!}{j!}t^{2n+2j+1}\end{aligned} \tag{1.78}$$

Now we can start to match coefficients of t‘s. On top, $(4n+1)$ terms corresponds to d_n‘s and $(4n-1)$ terms are 0. In summation, observe that first $n = 1, 2, \ldots, k$ term contains t^{2k+1} and each one has single contribution. So by letting $2k+1 = 2n+2j+1 \iff j = k-n$;

$$\sum_{n=0}^{k} d_n \frac{2^{2n}}{(2n)!}(-1)^{k-n}\frac{(k+n)!}{(k-n)!} \tag{1.79}$$

is the coefficient of t^{2k+1}. It follows;

$$\sum_{n=0}^{k} d_n \frac{2^{2n}}{(2n)!}(-1)^{k-n}\frac{(k+n)!}{(k-n)!} = \begin{cases} d_{k/2} & \text{if } k \text{ is even.} \\ 0 & \text{if } k \text{ is odd.} \end{cases} \tag{1.80}$$

To show explicitly, lets write out $k = 0, 1, 2, 3$ cases and I will denote each equation as $[0], [1], \ldots, [k], \ldots$.

$$\begin{aligned} &[0], \quad d_0 = d_0 \\ &[1], \quad d_0 - d_1 \frac{2^2}{2!}\frac{2!}{0!} = 0 \\ &[2], \quad d_0 - d_1 \frac{2^2}{2!}\frac{3!}{1!} + d_2 \frac{2^4}{4!}\frac{4!}{0!} = d_1 \\ &[3], \quad d_0 - d_1 \frac{2^2}{2!}\frac{4!}{2!} + d_2 \frac{2^4}{4!}\frac{5!}{1!} + d_3 \frac{2^6}{6!}\frac{6!}{0!} = 0 \quad \textit{etc.} \end{aligned} \tag{1.81}$$

We want to transform those equations to simpler ones. As Gauss did, look at $k^2[k] - (k-1)^2[k-2]$;

$$\begin{aligned} &= \sum_{n=0}^{k} d_n \frac{2^{2n}}{(2n)!}(-1)^{k-n} k^2 (k+n)(k+n-1)\cdots(k-n+1) \\ &\qquad - \sum_{n=0}^{k-2} d_n \frac{2^{2n}}{(2n)!}(-1)^{k-n}(k-1)^2(k+n-2)(k+n-3)\cdots(k-n-1) \end{aligned} \tag{1.82}$$

We first manipulate up to $k-2$ terms;

$$\sum_{n=0}^{k-2} d_n \frac{2^{2n}}{(2n)!}(-1)^{k-n}\left\{k^2(k+n)\cdots(k-n+1) - (k-1)^2(k+n-2)\cdots(k-n-1)\right\} \tag{1.83}$$

Consider only terms in curly brackets;

$$\begin{aligned} &k^2(k+n-2)\cdots(k-n-1)\left[\frac{(k+n)(k+n-1)}{(k-n-1)(k-n)} - 1\right] \\ &\qquad + (2k-1)(k+n-2)\cdots(k-n-1) \\ &= (2k-1)\left[(k+n-2)\cdots(k-n+1)\left(k^2 2n + (k-n)(k-n-1)\right)\right] \\ &= (2k-1)\left[(k+n-2)\cdots(k-n+1)\left((2n+1)k^2 - (2n+1)k + n(n+1)\right)\right] \end{aligned} \tag{1.84}$$

Observe;

$$(k+n-1)(k-n) = k^2 - k - n(n-1) \tag{1.85}$$

And rewrite as;

$$(2k-1)\left[(2n+1)(k+n-1)\cdots(k-n) + (k+n-2)\cdots(k-n+1)2n^3\right] \tag{1.86}$$

Turn back to (1.82) and split sum as;

$$\begin{aligned} &(2k-1)\sum_{n=0}^{k-2}(2n+1)d_n \frac{2^{2n}}{(2n)!}(-1)^{k-n}(k+n-1)\cdots(k-n) \\ &\qquad + (2k-1)\sum_{n=0}^{k-2} d_n \frac{2^{2n}}{(2n)!}(-1)^{k-n}(k+n-2)\cdots(k-n+1)2n^3 \end{aligned} \tag{1.87}$$

Second summation can start from $n = 1$ or as;

$$\begin{aligned}(2k-1)\sum_{n=0}^{k-2}(2n+1)d_n\frac{2^{2n}}{(2n)!}(-1)^{k-n}(k+n-1)\cdots(k-n)\\ +(2k-1)\sum_{n=0}^{k-3}d_{n+1}\frac{2^{2n+2}}{(2n+2)!}(-1)^{k-n-1}(k+n-1)\cdots(k-n)2(n+1)^3\end{aligned} \tag{1.88}$$

Leaving out $n = k - 2$ term;

$$(2k-1)\sum_{n=0}^{k-3}\frac{2^{2n}}{(2n+1)!}(-1)^{k-n}(k+n-1)\cdots(k-n)\left((2n+1)^2d_n-(2n+2)^2d_{n+1}\right) \tag{1.89}$$

Recall from (1.82) we left $n = k-1, k$ terms and from (1.88) we left $n = k-2$. Collect them together;

$$d_k2^{2k}k^2-(2k-1)d_{k-1}2^{2k-1}k^2+(2k-1)(2k-3)^2d_{k-2}2^{2k-4} \tag{1.90}$$

But this is exactly equal to sum of terms $n = k-2, k-1$ in (1.89)(one should do the calculation) therefore;

$$\begin{aligned}&k^2[k]-(k-1)^2[k-2];\\ &(2k-1)\sum_{n=0}^{k-1}\frac{2^{2n}}{(2n+1)!}(-1)^{k-n}(k+n-1)\cdots(k-n)\left[(2n+1)^2d_n-(2n+2)^2d_{n+1}\right]\end{aligned} \tag{1.91}$$

Last thing to find is what this sum equals to. There are 2 cases, if k is odd then it is equal to 0. If k is even it is equal to;

$$k^2d_{k/2}-(k-1)^2d_{(k-2)/2} \tag{1.92}$$

Let $k = 2m + 2$;

$$(2m+2)^2d_{m+1}-(2m+1)^2d_m \tag{1.93}$$

which is exactly in the form in summation of (1.91) and $m < k$. But we also know when $k = 1$, (1.91) reads;

$$0 = d_0 - 4d_1 \tag{1.94}$$

and hence it follows that (1.80) turns into;

$$0=d_0-4d_1=9d_1-16d_2=25d_2-36d_3=\cdots=(2k+1)^2d_k-(2k+2)^2d_{k+1}=\cdots \tag{1.95}$$

From that it is immediate;

$$d_k=\left(\frac{(2k-1)(2k-3)\cdots(2k+1-2j)}{(2k)(2k-2)\cdots(2k+2-2j)}\right)^2d_{k-j} \tag{1.96}$$

We know $d_0 = 1$;

$$d_k=\left(\frac{(2k-1)(2k-3)\cdots(1)}{(2k)(2k-2)\cdots(2)}\right)^2=\frac{1}{4^{2k-1}}\left[\frac{(2k-1)!}{k!(k-1)!}\right]^2 \tag{1.97}$$

Recall from (1.25) we know series representation of complete elliptic integral of first kind which is equal to $(M(1+k,1-k))^{-1}$ hence proof is done.

2. AGM in Complex Variables

In this section we will define agm in complex variables but it cannot be generalized directly because geometric mean is multi-valued function for complex variables. We will well-define agm, show convergence and at the end, state a main theorem for values of agm without proof.

For given a and b, there are uncountably many $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$. Also convergence is not obvious for any of them. First we restrict $a \neq \pm b$, $a, b \neq 0$. In those cases, convergence is clear and not interesting. And define a way to distinguish between two possible choices for each b_{n+1}.

Definition 2. Let $a, b \in \mathbb{C}^*$ satisfy $a \neq \pm b$. Then a square root b_1 of ab is called the *right choice* if $|a_1 - b_1| \leq |a_1 + b_1|$ and, when $|a_1 - b_1| = |a_1 + b_1|$, we also have $\mathrm{Im}(b_1/a_1) > 0$.

Lets see the case $\mathrm{Im}(b_1/a_1) = 0$. Then $b_1/a_1 = r \in \mathbb{R}$.

$$|a_1 - b_1| = |a_1||1-r| \neq |a_1||1+r| = |a_1 + b_1| \tag{2.1}$$

A natural way to define the agm as b_{n+1} always the right choice. However this is not the only possibility. To be precise;

Definition 3. Let $a, b \in \mathbb{C}^*$ satisfy $a \neq \pm b$. A pair of sequences $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ is called *good* if b_{n+1} is the right choice for all but finitely many $n \geq 0$.

It is expected to have this results after the definition:

Theorem 2. *If $a, b \in \mathbb{C}^*$ satisfy $a \neq \pm b$, then any pair of sequences $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ converge to a common limit, and this common limit is non-zero if and only if $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ are good sequences.*

Proof. Let $0 \leq \mathrm{ang}(a, b) \leq \pi$ denote the unoriented angle between a and b. For right choice b_1 we have;

$$|a_1 - b_1| \leq \frac{1}{2}|a - b| \tag{2.2}$$

$$\mathrm{ang}(a_1, b_1) \leq \frac{1}{2}\mathrm{ang}(a, b) \tag{2.3}$$

From $|a_1 - b_1| \leq |a_1 + b_1|$ observe;

$$|a_1 - b_1|^2 \leq |a_1 - b_1||a_1 + b_1| = \frac{1}{4}|a - b|^2 \tag{2.4}$$

Hence (2.2) is shown. Now let $\theta_1 = \mathrm{ang}(a_1, b_1)$ and $\theta = \mathrm{ang}(a, b)$. From the law of cosines;

$$|a_1 \pm b_1|^2 = |a_1|^2 + |b_1|^2 \pm 2|a_1||b_1|\cos\theta_1 \tag{2.5}$$

Rewrite $|a_1 - b_1| \leq |a_1 + b_1|$ from (2.5):

$$\begin{gathered} |a_1|^2 + |b_1|^2 - 2|a_1||b_1|\cos\theta_1 \leq |a_1|^2 + |b_1|^2 + 2|a_1||b_1|\cos\theta_1 \\ \iff \theta_1 \leq \pi/2 \end{gathered} \tag{2.6}$$

Then observe;

$$\mathrm{ang}(a_1, b_1) = \theta_1 \leq \pi - \theta_1 = \mathrm{ang}(a_1, -b_1) \tag{2.7}$$

To compare this to θ, note that one of $\pm b_1$, say b_1', satisfies $\mathrm{ang}(a, b_1') = \mathrm{ang}(b_1', b) = \theta/2$. This is because;

$$\begin{gathered} \text{Let } a = r_1 e^{i\theta_1} \text{ and } b = r_2 e^{i\theta_2} \\ b_1 = \begin{cases} \sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2} \\ \sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2+\pi)} \end{cases} \end{gathered} \tag{2.8}$$

And also noting that a_1 is in between a and b,

$$\mathrm{ang}(a_1, b_1) \leq \mathrm{ang}(a_1, b_1') \leq \frac{1}{2}\mathrm{ang}(a, b) \tag{2.9}$$

□

Now suppose $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ are not good sequences. Set $M_n = \max\{|a_n|, |b_n|\}$. Note that $M_{n+1} \leq M_n$. Recall definition of right choice and observe if b_{n+1} is not the right choice then;

$$|a_{n+1} + b_{n+1}| \leq |a_{n+1} - b_{n+1}| \iff |a_{n+2}| \leq \frac{1}{2}|a_{n+1} - b_{n+1}| \tag{2.10}$$

Apply (2.2);

$$|a_{n+2}| \leq \frac{1}{4}|a_n - b_n| \leq \frac{1}{2}M_n \tag{2.11}$$

We also have $|b_{n+2}| \leq M_n$. It follows;

$$\begin{aligned} M_{n+3} &= \max\{2^{-1}|a_{n+2} + b_{n+2}|, |\sqrt{a_{n+2}b_{n+2}}|\} \\ &\leq \max\{2^{-1}|(1/2)M_n + M_n|, |\sqrt{(1/2)M_n M_n}|\} \\ &\leq (3/4)M_n \end{aligned} \tag{2.12}$$

Since $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are not good sequences, (2.12) must occur infinitely often, proving that $\lim_{n\to\infty} M_n = 0$.

Now lets turn back to case $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are good sequences. Neglect first N terms where N is sufficiently large such that b_{n+1} is always right choice for $n \geq N$. Then say b_{n+1} is always right choice for $n \geq 0$. Also $\operatorname{ang}(a,b) < \pi$ follows from (2.3). Set $\theta_n = \operatorname{ang}(a_n, b_n)$. From (2.2) and (2.3);

$$|a_n - b_n| \leq 2^{-n}|a-b|, \quad \theta_n \leq 2^{-n}\theta \tag{2.13}$$

Note that $a_n - a_{n+1} = (1/2)(a_n - b_n)$, so by (2.13);

$$|a_n - a_{n+1}| \leq 2^{n+1}|a-b| \tag{2.14}$$

If $m > n$;

$$|a_n - a_m| \leq \sum_{k=n}^{m-1} |a_k - a_{k+1}| \leq \left(\sum_{k=n}^{m-1} 2^{-(k+1)}\right)|a-b| < 2^{-n}|a-b| \tag{2.15}$$

Thus $\{a_n\}_{n=0}^{\infty}$ converges because it is a Cauchy sequence, and then by (2.13) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Lastly, we need to show that common limit is nonzero. Let;

$$m_n = \min\{|a_n|, |b_n|\} \tag{2.16}$$

Observe that $|b_{n+1}| = |\sqrt{a_n b_n}| \geq m_n$. To relate $|a_{n+1}|$ and m_n, use the law of cosines;

$$\begin{aligned} (2|a_{n+1}|)^2 &= |a_n|^2 + |b_n|^2 + 2|a_n||b_n|\cos\theta_n \\ &\geq 2m_n^2(1+\cos\theta_n) = 4m_n^2\cos^2(\theta_n/2) \end{aligned} \tag{2.17}$$

Then it follows;

$$\begin{aligned} m_{n+1} &= \min\{|a_{n+1}|, |b_{n+1}|\} \geq \min\{m_n\cos(\theta_n/2), m_n\} \\ m_{n+1} &\geq \cos(\theta_n/2)m_n \\ m_{n+1} &\geq \end{aligned} \tag{2.18}$$

From (2.13) we can write(recall $\operatorname{ang}(a,b) < \pi$);

$$\begin{aligned} m_{n+1} \geq \cos(\theta_n/2)m_n &\geq \cos(\theta/2^{n+1})m_n \\ m_{n+1} &\geq \prod_{k=1}^{n+1}\cos(\theta/2^k)m_0 \end{aligned} \tag{2.19}$$

Lets show well known equality;

$$\begin{aligned} \sin\theta = 2\cos(\theta/2)\sin(\theta/2) &= 4\cos(\theta/2)\cos(\theta/4)\sin(\theta/4) \\ &= \cdots = 2^n\sin(\theta/2^n)\prod_{k=1}^{n}\cos(\theta/2^k) \end{aligned} \tag{2.20}$$

Therefore;

$$\sin\theta = \lim_{n\to\infty} 2^n\sin(\theta/2^n)\prod_{k=1}^{n}\cos(\theta/2^k) = \theta\prod_{k=1}^{\infty}\cos(\theta/2^k) \tag{2.21}$$

This proves;

$$\frac{\sin\theta}{\theta} = \prod_{k=1}^{\infty}\cos(\theta/2^k) \tag{2.22}$$

From this rewrite (2.19);

$$m_n \geq \frac{\sin\theta}{\theta}m_0 \tag{2.23}$$

which proves that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \neq 0$.

Definition 4. Let $a, b \in \mathbb{C}^*$ satisfy $a \neq \pm b$. A nonzero complex number μ is a value of the *arithmetic-geometric* mean M(a,b) of a and b if there are good sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ such that

$$\mu = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \tag{2.24}$$

Hence $M(a,b)$ is a multiple valued function of a and b and there are countable number of values. Note that if all choices are good choice for b_{n+1} then the common limit is called *simplest value* of $M(a,b)$. Which is also equivalent to definition in ysection 1 when $a,b\in\mathbb{R}^+$.

3. Theta Series Solution to the AGM

The basic *theta functions* are defined for $|q|<1$ by;

$$\theta_2(q)\equiv\sum_{n=-\infty}^{\infty}q^{(n+1/2)^2},\quad \theta_2(0)=0 \tag{3.1}$$

$$\theta_3(q)\equiv\sum_{n=-\infty}^{\infty}q^{n^2},\quad \theta_3(0)=1 \tag{3.2}$$

$$\theta_4(q)\equiv\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2},\quad \theta_4(0)=1 \tag{3.3}$$

Observe $\theta_4(q)=\theta_3(-q)$ and also;

$$\theta_3(q)+\theta_4(q)=2\sum_{n\text{ even}}q^{n^2}=2\sum_{n=-\infty}^{\infty}q^{4n^2}=2\theta_3(q^4) \tag{3.4}$$

and

$$\theta_3(q)-\theta_4(q)=2\sum_{n\text{ odd}}q^{n^2}=2\sum_{n=-\infty}^{\infty}q^{4(n+1/2)^2}=2\theta_2(q^4) \tag{3.5}$$

Represent;

$$\theta_3^2(q)=\sum_{n=0}^{\infty}r_2(n)q^n\qquad \theta_4^2(q)=\sum_{n=0}^{\infty}(-1)^nr_2(n)q^n \tag{3.6}$$

where $r_2(n)$ counts the number of ways of writing $n=j^2+k^2$ and distinguish sign and permutation. Ex. $r_2(5)=8$. Set $r_2(0)=1$. For $\theta_4^2(q)$, examine cases when n is even and odd and what it implies about its representation as j^2+k^2. It is also useful to represent $\theta_3^2(q)$ and $\theta_4^2(q)$(or any other) as summation over even and odd integers seperately;

$$\begin{aligned}\theta_3^2(q)&=\sum_{\substack{n\geq 0\\ \text{n even}}}^{\infty}r_2(n)q^n+\sum_{\substack{n\geq 0\\ \text{n odd}}}^{\infty}r_2(n)q^n=\sum_{n=0}^{\infty}r_2(2n)q^{2n}+\sum_{n=0}^{\infty}r(2n+1)q^{2n+1}\\ \theta_4^2(q)&=\sum_{\substack{n\geq 0\\ \text{n even}}}^{\infty}r_2(n)q^n-\sum_{\substack{n\geq 0\\ \text{n odd}}}^{\infty}r_2(n)q^n=\sum_{n=0}^{\infty}r_2(2n)q^{2n}-\sum_{n=0}^{\infty}r(2n+1)q^{2n+1}\end{aligned} \tag{3.7}$$

Now to observe $r_2(2n)=r_2(n)$, consider one such representation of n as $n=j^2+k^2$;

$$2n=2j^2+2k^2=(j+k)^2+(j-k)^2 \tag{3.8}$$

So any representation of n is also representing $2n$ in the form (3.8). Avoidable but letting $2n=a^2+b^2$, by writing $a=j+k, b=j-k$ one can solve it uniquely to determine j and k. This leds to;

$$\theta_3^2(q)+\theta_4^2(q)=2\sum_{n=0}^{\infty}r_2(2n)q^{2n}=2\theta_3^2(q^2) \tag{3.9}$$

Now, (3.4) and (3.9) allow us to solve $\theta_3(q)\theta_4(q)$;

$$\begin{aligned}\theta_3(q)\theta_4(q)&=\frac{1}{2}[\theta_3(q)+\theta_4(q)]^2-\frac{1}{2}[\theta_3^2(q)+\theta_4^2(q)]\\ &=2\theta_3^2(q^4)-\theta_3^2(q^2)=\theta_4^2(q^2)\end{aligned} \tag{3.10}$$

To observe last equality;

$$
\begin{aligned}
2\theta_3^2(q^4) - \theta_3^2(q^2) &= 2\sum_{n=0}^{\infty} r_2(n)q^{4n} - \sum_{n=0}^{\infty} r_2(2n)q^{4n} - \sum_{\substack{n\geq 0 \\ \text{n odd}}}^{\infty} r_2(n)q^{2n} \\
&= \sum_{\substack{n\geq 0 \\ \text{n even}}}^{\infty} r_2(n)q^{2n} - \sum_{\substack{n\geq 0 \\ \text{n odd}}}^{\infty} r_2(n)q^{2n} = \theta_4^2(q^2)
\end{aligned} \tag{3.11}
$$

Thus;

$$
\begin{aligned}
\frac{\theta_3^2(q) + \theta_4^2(q)}{2} &= \theta_3^2(q^2) \\
\sqrt{\theta_3^2(q)\theta_4^2(q)} &= \theta_4^2(q^2)
\end{aligned} \tag{3.12}
$$

which bears an obvious resemblance to the AGM. Similarly;

$$
\theta_3^2(q) - \theta_3^2(q^2) = \sum_{n=0}^{\infty} r_2(2n)q^{2n} + \sum_{\substack{n\geq 0 \\ \text{n odd}}}^{\infty} r_2(n)q^{n} - \sum_{n=0}^{\infty} r_2(n)q^{2n} = \sum_{\substack{n\geq 0 \\ \text{n odd}}}^{\infty} r_2(n)q^{n} \tag{3.13}
$$

Last term can be written as;

$$
\sum_{\substack{n\geq 0 \\ \text{n odd}}}^{\infty} r(n)q^n = \sum_{\substack{k,m=-\infty \\ k+m \text{ odd}}}^{\infty} q^{m^2+k^2} \tag{3.14}
$$

$k+m$ odd is equivalent to $k+m = 2i+1$ hence let $k = i-j$ and $m = i+j+1$. As a side note, summation over i will cover all odd integers and summation over j for some i covers all possible representations of the same $k+m$. More formally;

$$
= \sum_{i=-\infty}^{\infty} \left(\sum_{\substack{k,m=-\infty \\ k+m=2i+1}}^{\infty} q^{m^2+k^2} \right) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} q^{(i+j+1)^2+(i-j)^2} = \sum_{i,j=-\infty}^{\infty} (q^2)^{(i+1/2)^2+(j+1/2)^2} = \theta_2^2(q^2) \tag{3.15}
$$

Hence;

$$
\theta_3^2(q^2) + \theta_2^2(q^2) = \theta_3^2(q) \tag{3.16}
$$

Combine this with first line of (3.12);

$$
\theta_3^2(q^2) - \theta_2^2(q^2) = \theta_4^2(q) \tag{3.17}
$$

Last two and second line of (3.12) yields Jacobi's Identity as follows;

$$
4\theta_2^4(q^2) = \theta_3^4(q) + \theta_4^4(q) - 2\theta_3^2(q)\theta_4^2(q) \tag{3.18}
$$

But;

$$
\theta_3^4(q) + \theta_4^4(q) = 2\theta_3^4(q^2) + 2\theta_2^4(q^2) \tag{3.19}
$$

Combine to get;

$$
2\theta_2^4(q^2) = 2\theta_3^4(q^2) - 2\theta_4^4(q^2) \tag{3.20}
$$

Rewrite in the form of Jacobi's Identity;

$$
\theta_3^4(q) = \theta_2^4(q) + \theta_4^4(q) \tag{3.21}
$$

Now set $k \equiv k(q) \equiv \theta_2^2(q)/\theta_3^2(q)$. Then (3.21) shows $k' = \sqrt{1-k^2} = \theta_4^2(q)/\theta_3^2(q)$. Return to (3.12) and set $a_n = \theta_3^2(q^{2^n})$ and $b_n = \theta_4^2(q^{2^n})$. Observe a_n and b_n satisfy the AGM iteration. Moreover, because we know the common limit as $n \to \infty$;

$$
M(\theta_3^2(q), \theta_4^2(q)) = 1 \tag{3.22}
$$

Stating our observations as Theorem;

Theorem 3. *Let $0 < k < 1$ be given. The AGM satisfies;*

$$M(1,k') = \theta_3^{-2}(q) \qquad \textit{for} \qquad k' = \theta_4^2(q)/\theta_3^2(q) \tag{3.23}$$

where q is the unique solution in $(0,1)$ to $k = \theta_2^2(q)/\theta_3^2(q)$. In particular;

$$K(k) = \frac{\pi}{2}\theta_3^2(q) \tag{3.24}$$

3.1. **Rederive Jacobi's Identity.**

First of all, establish a formal identity for $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}$;

$$\sum_{m,n=-\infty}^{\infty} f(m,n) = \sum_{l,k=-\infty}^{\infty} f(l+k, l-k) + \sum_{l,k=-\infty}^{\infty} f(l+k, l-k-1) \tag{3.25}$$

Proof. Fix any integer pair (a,b). Observe that if both of them have the same parity then;

$$l = \frac{a+b}{2}, \qquad k = \frac{a-b}{2} \tag{3.26}$$

has integer solution. While if their parities are different;

$$l = \frac{a+b+1}{2}, \qquad k = \frac{a-b-1}{2} \tag{3.27}$$

has integer solution. Therefore observe that summations are over disjoint sets and they cover $\mathbb{Z}\times\mathbb{Z}$. □

Now apply (3.25) to $q^{m^2+n^2}$;

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty} q^{m^2+n^2} &= \sum_{l,k=-\infty}^{\infty} q^{(l+k)^2+(l-k)^2} + \sum_{l,k=-\infty}^{\infty} q^{(l+k)^2+(l-k-1)^2} \\ &= \sum_{l,k=-\infty}^{\infty} q^{2(l^2+k^2)} + \sum_{l,k=-\infty}^{\infty} q^{2\{(l-\frac{1}{2})^2+(k+\frac{1}{2})^2\}}\end{aligned} \tag{3.28}$$

This is (3.16). Similarly apply (3.25) to $(-1)^{m+n}q^{m^2+n^2}$;

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2} &= \sum_{l,k=-\infty}^{\infty} q^{(l+k)^2+(l-k)^2} - \sum_{l,k=-\infty}^{\infty} q^{(l+k)^2+(l-k-1)^2} \\ &= \sum_{l,k=-\infty}^{\infty} q^{2(l^2+k^2)} - \sum_{l,k=-\infty}^{\infty} q^{2\{(l-\frac{1}{2})^2+(k+\frac{1}{2})^2\}}\end{aligned} \tag{3.29}$$

This is (3.17). Lastly apply it to $(-1)^m q^{m^2+n^2}$;

$$\sum_{m,n=-\infty}^{\infty} (-1)^m q^{m^2+n^2} = \sum_{l,k=-\infty}^{\infty} (-1)^{l+k} q^{2(l^2+k^2)} + \sum_{l,k=-\infty}^{\infty} (-1)^{l+k} q^{(l+k)^2+(l-k-1)^2} \tag{3.30}$$

Notice that power of q at the last sum is symmetric under $k \to -k-1$, but both appear with opposite sign, therefore it is 0 and observing that it is the second line of (3.12).

Note 2. By knowing (3.16) and (3.17), it is straightforward to get the first line of (3.12).

Note 3. Jacobi's Identity follows exactly as discussed after (3.17) and hence by (3.25) we achived all required identities to prove (3).

3.2. **The Poisson Summation Formula.**

Theorem 4. *Let f be a nonnegative function such that the integral $\int_{-\infty}^{\infty} f$ exists as an improper Riemann Integral. Assume also that f increases on $(-\infty, 0]$ and decreases on $[0,\infty)$. Then we have;*

$$\sum_{m=-\infty}^{\infty} \frac{f(m+)+f(m-)}{2} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt \tag{3.31}$$

each series being absolutely convergent.

Proof. Proof makes use of the Fourier expansion of the function F defined by the series

$$F(x) = \sum_{m=-\infty}^{\infty} f(m+x) \tag{3.32}$$

First we show that this series converges absolutely for each real x and the convergence is uniform on the interval $[0,1]$.

We will use Weierstrass M-test, which is stated as;

Theorem 5. *Let $\{M_n\}$ be a sequence of nonnegative numbers such that*

$$|f_n(x)| \leq M_n \qquad \text{for } n = 1, 2, \ldots \text{ and for every } x \text{ in } S. \tag{3.33}$$

Then $\sum f_n(x)$ converges uniformly on S if $\sum M_n$ converges.

Let $f_n(x) \equiv f(n+x)$ and for $x \geq 0$, $n > 0$;

$$\begin{aligned} |f_0(x)| &\leq f(0) \equiv M_0 \\ |f_n(x)| &\leq f(n) \leq \int_{n-1}^{n} f(t)dt \equiv M_n \end{aligned} \tag{3.34}$$

We assumed $\sum_{n=0}^{\infty} M_n$ converges hence by Theorem (5) $\sum_{n=0}^{\infty} f(x+n)$ converges uniformly on $[0,\infty)$.

Similarly for $x \leq 1$, $n < -1$;

$$\begin{aligned} |f_{-1}(x)| &\leq f(0) \equiv M_{-1} \\ |f_n(x)| &\leq f(n+1) \leq \int_{n+1}^{n+2} f(t)dt \equiv M_n \end{aligned} \tag{3.35}$$

Now $\sum_{n=-1}^{-\infty} M_n$ converges hence $\sum_{n=-1}^{-\infty} f(x+n)$ converges uniformly on $(-\infty, 1]$. Therefore the series in (3.32) converges for all x and the convergence is uniform on the intersection $[0,1]$.

Observe that F is periodic with period 1. In fact, we have $F(x+1) = \sum_{m=-\infty}^{\infty} f(m+x+1)$, and this series is merely a rearrangement of that in (3.32). Since all its terms are nonnegative, it converges to the same sum. Hence;

$$F(x+1) = F(x) \tag{3.36}$$

Observe if $0 \leq x \leq \frac{1}{2}$, then $f(m+x)$ is a decreasing function of x if $m \geq 0$, and an increasing function of x if $m < 0$. Therefore we have;

$$F(x) = \sum_{m=0}^{\infty} f(m+x) - \sum_{m=-\infty}^{-1} \{-f(m+x)\} \tag{3.37}$$

so F is the difference of two decreasing functions. Similarly, if $-\frac{1}{2} \leq x \leq 0$, then $f(m+x)$ is decreasing function of x if $m \geq 1$ and an increasing function of x if $m < 1$ and can be written similar to (3.37). Therefore F is of bounded variation on $[-\frac{1}{2}, \frac{1}{2}]$. By periodicity, F is of bounded variation on every compact interval.

Now consider the Fourier series(in exponential form) generated by F, say;

$$F(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi i n x} \tag{3.38}$$

Since F is of bounded variation on $[0,1]$ it is Riemann-integrable on $[0,1]$ and the Fourier coefficients are given by the formula;

$$\alpha_n = \int_0^1 F(x) e^{-2\pi i n x} dx \tag{3.39}$$

Also, since F is of bounded variation on every compact interval, Jordan's test shows that the Fourier series converges for every x and that

$$\frac{F(x+) + F(x-)}{2} = \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi i n x} \tag{3.40}$$

By using (3.32), integrate (3.39) term by term. *(Justified by Uniform Convergence).*

$$\alpha_n = \sum_{m=-\infty}^{\infty} \int_0^1 f(m+x)e^{-2\pi inx}dx \tag{3.41}$$

Make change of variable $t = m + x$;

$$\alpha_n = \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(t)e^{-2\pi int}dt = \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt \tag{3.42}$$

Using this in (3.40);

$$\frac{F(x+)+F(x-)}{2} = \sum_{n=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt \right\} e^{2\pi inx} \tag{3.43}$$

Which is *The Poisson Summation Formula* when $x = 0$. □

Example 1. Apply (3.43) to;

$$f(x) := \begin{cases} e^{-yx} & x \geq 0, \\ 0 & x < 0, \end{cases} \quad y > 0. \tag{3.44}$$

The right-hand side becomes;

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi inx}}{y+2\pi in} &= \frac{1}{y} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}(y-2\pi in)}{y^2+(2\pi n)^2} \\ &= \frac{1}{y} + 2\sum_{n=1}^{\infty} \frac{y\cos(2\pi nx) + (2\pi n)\sin(2\pi nx)}{y^2+(2\pi n)^2} \end{aligned} \tag{3.45}$$

and for left-hand side, note that F is not continuous at $x \in \mathbb{Z}$;

$$\sum_{n>-x} e^{-y(n+x)} + \begin{cases} \frac{1}{2} & x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{3.46}$$

If $x := 0$;

$$\frac{1}{2} + \frac{1}{e^y - 1} = \frac{1}{y} + 2y\sum_{n=1}^{\infty} \frac{1}{y^2+(2\pi n)^2} \tag{3.47}$$

Recall definition of *Hyperbolic cotangent*;

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = 1 + \frac{2}{e^{2x}-1} \tag{3.48}$$

hence;

$$\frac{1}{2}\coth\left(\frac{x}{2}\right) = \frac{1}{2} + \frac{1}{e^x - 1} \tag{3.49}$$

Or;

$$\begin{aligned} \frac{1}{2}\coth(\pi y) &= \frac{1}{2} + \frac{1}{e^{2\pi y}-1} \\ &= \frac{1}{2\pi y} + \frac{y}{\pi}\sum_{n=1}^{\infty}\frac{1}{y^2+n^2} \end{aligned} \tag{3.50}$$

Therefore we obtain classical formula for coth;

$$\pi\coth(\pi x) = \frac{1}{x} + 2x\sum_{n=1}^{\infty}\frac{1}{x^2+n^2} \tag{3.51}$$

If $x := \frac{1}{2}$, notice that on right-hand side summation becomes alternating and left-hand side becomes;

$$\frac{1}{e^{y/2} - e^{-y/2}} = \frac{1}{2}\operatorname{csch}\left(\frac{y}{2}\right) \tag{3.52}$$

and exact same change of variable for coth can be applied to get;

$$\pi \operatorname{csch}(\pi x) = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2+n^2} \tag{3.53}$$

Example 2. Now apply (3.43) to;

$$f(x) := e^{-sx^2\pi} \qquad s > 0. \tag{3.54}$$

Then;

$$\sum_{n=-\infty}^{\infty} e^{-s(n+x)^2\pi} = \sum_{k=-\infty}^{\infty} e^{2\pi ikx} \int_{-\infty}^{\infty} e^{-st^2\pi - 2\pi ikt} dt \tag{3.55}$$

Integral on the right is;

$$\begin{aligned} 2\int_0^{\infty} e^{-s\pi t^2} \cos(2\pi kt)dt &= 2\int_0^{\infty} e^{-x^2} \cos\left(2\sqrt{\frac{\pi}{s}}kx\right) \frac{dx}{\sqrt{s\pi}} \\ &= \frac{2}{\sqrt{s\pi}} F\left(\sqrt{\frac{\pi}{s}}k\right) \end{aligned} \tag{3.56}$$

where;

$$F(y) := \int_0^{\infty} e^{-x^2} \cos(2xy)dx = \frac{\sqrt{\pi}}{2} e^{-y^2} \tag{3.57}$$

For the last equality[2];

$$\begin{aligned} \dot{F}(y) &= -\int_0^{\infty} 2xe^{-x^2} \sin(2xy)dx \\ &= -2y\int_0^{\infty} e^{-x^2} cos(2xy)dx \end{aligned} \tag{3.58}$$

Therefore F satisfies;

$$\begin{cases} \dot{F}(y) + 2yF(y) & = 0 \\ F(0) & = \frac{\sqrt{\pi}}{2} \end{cases} \tag{3.59}$$

Thus we get;

$$\sum_{n=-\infty}^{\infty} e^{-s(n+x)^2\pi} = \frac{1}{\sqrt{s}} \sum_{k=-\infty}^{\infty} e^{2\pi ikx} e^{-\frac{\pi k^2}{s}} \tag{3.60}$$

This is a general form of the *theta transformation formula* which holds for $\mathrm{re}(s) > 0$. For $x := 0$ (3.60) reduces to;

$$\sqrt{s}\theta_3(e^{-s\pi}) = \theta_3(e^{-\pi/s}) \tag{3.61}$$

3.3. **Poisson Summation and The AGM.** Recall setting $x := 0$ gives (3.61) while $x := \frac{1}{2}$ gives(and setting $s = s^{-1}$);

$$\begin{aligned} \sqrt{s}\theta_4(e^{-s\pi}) &= \theta_2(e^{-\pi/s}) \\ \sqrt{s}\theta_2(e^{-s\pi}) &= \theta_4(e^{-\pi/s}) \end{aligned} \tag{3.62}$$

Recall Theorem (3) and divide above equations to get;

$$k(e^{-s\pi}) = k'(e^{-\pi/s}) \tag{3.63}$$

Now from the theorem (3), setting $q := e^{-s\pi}$;

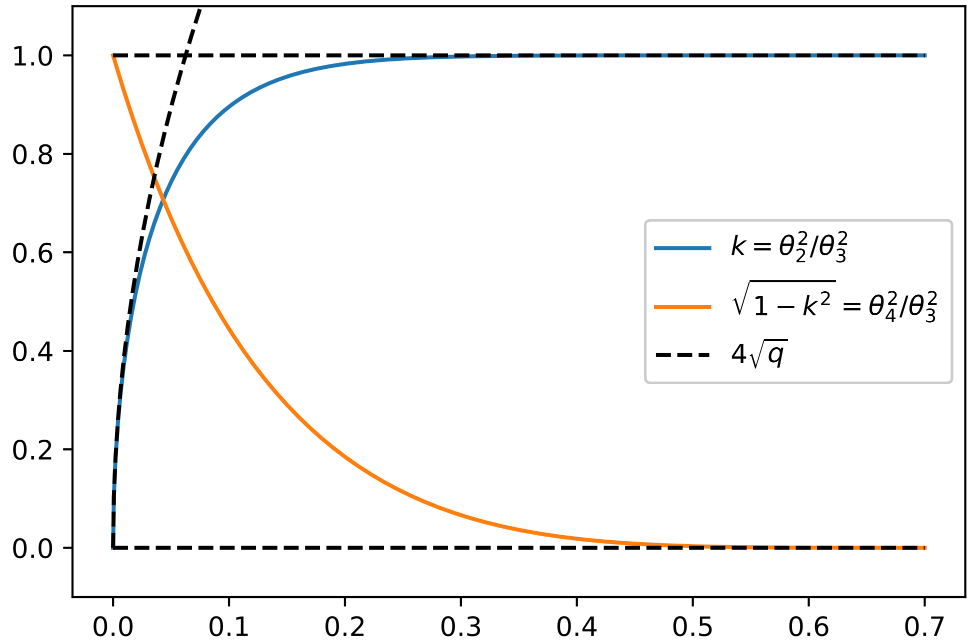
$$M(1,k) = \theta_3^{-2}(e^{-\pi/s}) \tag{3.64}$$

Thus;

$$\pi\frac{M(1,k')}{M(1,k)} = \pi\frac{\theta_3^2(e^{-\pi/s})}{\theta_3^2(e^{-s\pi})} = \pi s = -\log q \tag{3.65}$$

Conclusion of this discussion can be given as a fundamental theorem as follows;

[2]Justified by continuity of integrand and its derivative



Theorem 6. *For all k in $(0,1)$,*

$$\pi \frac{M(1,k')}{M(1,k)} = -\log q, \quad k = \frac{\theta_2^2(q)}{\theta_3^2(q)}, \quad k' = \frac{\theta_4^2(q)}{\theta_3^2(q)} \tag{3.66}$$

and so

$$\pi \frac{K'(k)}{K(k)} = -\log q \tag{3.67}$$

Note 4. Second equation is often written as $q = e^{-\pi K'/K}$ and q is called the *nome* associated with k. In principle it solves the inversion problem for q in terms of k.

Lets observe that $k = \theta_2^2(q)/\theta_3^2(q) = 4\sqrt{q}f(q)$ where $f(q)$ is analytic and $f(0) = 1$.

$$\frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\left[2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\right]^2}{\left[1 + 2\sum_{n=1}^{\infty} q^{n^2}\right]^2} = 4\sqrt{q}\left[\frac{\sum_{n=0}^{\infty} q^{n^2+n}}{1 + 2\sum_{n=1}^{\infty} q^{n^2}}\right]^2 \tag{3.68}$$

For asymptotic of $f(q)$ as $q \to 0$,

$$f(q) = \left[\frac{1 + O(q^2)}{1 + O(q)}\right]^2 = \left[(1 + O(q^2))(1 + O(q))\right]^2 = 1 + O(q) \tag{3.69}$$

Hence $k = 4\sqrt{q} + O(q)$. From this and Theorem (6), since $M(1,k') \to 1$ as $k \to 0^+$;

$$\lim_{k \to 0^+} \left[\frac{\pi}{2M(1,k)} - \log\left(\frac{4}{k}\right)\right] = 0 \tag{3.70}$$

Now consider the AGM iteration with $a_n(q) := \theta_3^2(q^{2^n})/\theta_3^2(q)$ and $b_n(q) := \theta_4^2(q^{2^n})/\theta_3^2(q)$. Notice this is equivalent to considering AGM iteration with $a_0 := 1$ and $b_0 := k'$. We know $b_n/a_n = k'(q^{2^n})$ and

$c_n/a_n = k(q^{2^n})$ but latter needs a verification as follows;

$$\frac{c_n}{a_n} = \frac{c_{n-1}^2}{4a_n^2} = \frac{(a_{n-2}-b_{n-2})^2}{16a_n^2} = \frac{\left(\theta_3^2(q^{2^{n-2}}) - \theta_4^2(q^{2^{n-2}})\right)^2}{16\theta_3^4(q^{2^n})} = \frac{\theta_2^2(q^{2^n})}{\theta_3^2(q^{2^n})} = k(q^{2^n}) \tag{3.71}$$

Above we used (3.4) and (3.5). Also it can be shown;

$$\log\left(\frac{4}{k(q^{2^n})}\right) = -2^{n-1}\log q + O(q^{2^n} - q^{2^{n-1}}) \tag{3.72}$$

Which leads to;

$$\lim_{n\to\infty} 2^{-n}\log\left(\frac{4a_n}{c_n}\right) = \frac{\pi}{2}\frac{M(1,k')}{M(1,k)} \tag{3.73}$$

From (3.65) and (3.73);

$$\lim_{n\to\infty} 2^{1-n}\log\left(\frac{4a_n}{c_n}\right) = \pi s \tag{3.74}$$

Define;

$$\pi_n := 2^{1-n}\frac{d}{ds}\log\left(\frac{a_n}{c_n}\right) \to \pi \tag{3.75}$$

Now state a lemma due to Gauss;

Lemma 1. $2^{-n}b_n^{-2}\boldsymbol{d}\log(a_n/c_n)$ *is independent of* n.

Using lemma;

$$\pi_n = \frac{b_n^2\pi_0}{b_0^2} \qquad \text{while} \qquad \pi_0 = -\frac{2}{k}\frac{dk}{ds} \tag{3.76}$$

Recall (1.24) and since b_n tends to $M(1,k')$;

$$\frac{dk}{ds} = -\frac{\pi}{2}\frac{kk'^2}{M(1,k')^2} = -\frac{2}{\pi}kk'^2K^2 \tag{3.77}$$

and since $q = e^{-\pi s}$;

$$\frac{dk}{dq} = \frac{1}{2q}\frac{kk'^2}{M(1,k')^2} = \frac{2kk'^2K^2}{q\pi^2} \tag{3.78}$$

Rewriting k, k' and K in (3.77) in theta terms produces;

$$\frac{\dot\theta_3}{\theta_3} - \frac{\dot\theta_2}{\theta_2} = \frac{\pi}{4}\theta_4^4 \qquad \text{(w.r.t.s)} \tag{3.79}$$

Differentiation of (3.61) and (3.62) yields;

$$\begin{aligned} s^2\dot\theta_3(e^{-\pi s}) + \frac{s}{2}\theta_3(e^{-\pi s}) &= -s^{-1/2}\dot\theta_3(e^{-\pi/s}) \\ s^2\dot\theta_4(e^{-\pi s}) + \frac{s}{2}\theta_4(e^{-\pi s}) &= -s^{-1/2}\dot\theta_2(e^{-\pi/s}) \end{aligned} \tag{3.80}$$

Again using (3.61) and (3.62);

$$\begin{aligned} s\frac{\dot\theta_3}{\theta_3}(e^{-\pi s}) + s^{-1}\frac{\dot\theta_3}{\theta_3}(e^{-\pi/s}) &= -\frac{1}{2} \\ s\frac{\dot\theta_4}{\theta_4}(e^{-\pi s}) + s^{-1}\frac{\dot\theta_2}{\theta_2}(e^{-\pi/s}) &= -\frac{1}{2} \end{aligned} \tag{3.81}$$

Substract to get;

$$s^2\left(\frac{\dot\theta_3}{\theta_3} - \frac{\dot\theta_2}{\theta_2}\right)(e^{-\pi s}) = \left(\frac{\dot\theta_4}{\theta_4} - \frac{\dot\theta_3}{\theta_3}\right)(e^{-\pi/s}) \tag{3.82}$$

Notice left-hand side is equivalent to (3.79);

$$\left(\frac{\dot\theta_4}{\theta_4} - \frac{\dot\theta_3}{\theta_3}\right) = \frac{\pi}{4}\theta_2^4 \tag{3.83}$$

Finally, add (3.83) and (3.79)(and use Jacobi's Identity);

$$\left(\frac{\dot{\theta}_4}{\theta_4} - \frac{\dot{\theta}_2}{\theta_2}\right) = \frac{\pi}{4}\theta_3^4 \tag{3.84}$$

Now we are ready to express E in terms of theta functions. Recall (1.41) and use (3.77);

$$E - K = kk'^2\frac{dK}{dk} - k^2K = -\frac{\pi}{K}\left[\frac{1}{2K}\frac{dK}{ds} + \frac{k^2K^2}{\pi}\right] \tag{3.85}$$

Express last expression in terms of theta functions;

$$E - K = -\frac{\pi}{K}\left[\frac{\dot{\theta}_3}{\theta_3} + \frac{\pi}{4}\theta_2^4\right] \tag{3.86}$$

Hence;

$$E = K - \frac{\pi}{K}\frac{\dot{\theta}_4}{\theta_4} \tag{3.87}$$

Similarly(Using (3.63));

$$E' = K' - \frac{\pi}{K'}\frac{\dot{\theta}_4}{\theta_4}(e^{-\pi/s}) \tag{3.88}$$

Now from (3.81) and noting $K'/K = s$ (from (6));

$$\frac{\dot{\theta}_4}{\theta_4}(e^{-\pi/s}) = -\frac{K'}{2K} - \frac{K'^2}{K^2}\frac{\dot{\theta}_2}{\theta_2} \tag{3.89}$$

Now rewrite (3.84) in terms of K;

$$\frac{\dot{\theta}_2}{\theta_2} = \frac{\dot{\theta}_4}{\theta_4} - \frac{K^2}{\pi} \tag{3.90}$$

Combining all will show;

$$E' = \frac{\pi}{2K} + \frac{\pi K'}{K^2}\frac{\dot{\theta}_4}{\theta_4} \tag{3.91}$$

4. The Derived Iteration and Some Convergence Results

Recall (1.41) and put $k = 1/\sqrt{2}$ to get;

$$\frac{dK}{dk}(1/\sqrt{2}) = 2\sqrt{2}\left[E(1/\sqrt{2}) - \frac{1}{2}K(1/\sqrt{2})\right] \tag{4.1}$$

It is straightforward to evaluate it from (A.8) and;

$$\dot{K}(1/\sqrt{2}) = \frac{2^{3/2}\pi^{3/2}}{\Gamma^2(1/4)} \tag{4.2}$$

Observing this our motivation is $\sqrt{2}K(1/\sqrt{2})\dot{K}(1/\sqrt{2}) = \pi$. Now consider the AGM with $a_0 := 1$ and $b_0 := k$. a_n and b_n viewed as functions of k converge uniformly and analytically to $M(1,k)$. It follows that the derived iterations $\dot{a}_n$ and $\dot{b}_n$ converge to $\dot{M}(1,k)$. Since $M(1,k) = \pi/2K'(k)$;

$$\dot{M}(1,k) = \frac{\pi}{2}\frac{d}{dk}\left(\frac{1}{K(k')}\right) = \frac{\pi}{2}\frac{d}{dk'}\left(\frac{1}{K(k')}\right)\frac{dk'}{dk} = \frac{\pi}{2}\frac{k}{k'}\frac{\dot{K}}{K^2}(k') \tag{4.3}$$

Or equivalently;

$$\dot{K}(k') = \frac{\pi}{2}\frac{k'}{k}\frac{\dot{M}(1,k)}{M^2(1,k)} \tag{4.4}$$

Now the derived iteration is

$$\dot{a}_{n+1} = \frac{\dot{a}_n + \dot{b}_n}{2} \qquad \dot{b}_{n+1} = \frac{\dot{a}_n\sqrt{b_n/a_n} + \dot{b}_n\sqrt{a_n/b_n}}{2} \tag{4.5}$$

For convergencence first observe;

$$\dot{b}_{n+1} \geq \dot{a}_{n+1} \iff \dot{a}_n\left(1 - \sqrt{b_n/a_n}\right) \leq \dot{b}_n\left(\sqrt{a_n/b_n} - 1\right) \iff \dot{b}_n \geq \dot{a}_n\sqrt{\frac{b_n}{a_n}} \tag{4.6}$$

Since $\dot{a}_0 = 0$, it is always true that $\dot{b}_n \geq \dot{a}_n$. It is left to show $\dot{a}_{n+1} \geq \dot{a}_n$ and $\dot{b}_{n+1} \leq \dot{b}_n$, and former is straightforward but latter requires an attention. But first we consider the *Legendre forms* $x_n := a_n/b_n$ and $y_n := \dot{b}_n/\dot{a}_n$. Then

$$x_{n+1} = \frac{\sqrt{x_n} + 1/\sqrt{x_n}}{2} \qquad y_{n+1} = \frac{y_n\sqrt{x_n} + 1/\sqrt{x_n}}{y_n + 1} \tag{4.7}$$

where $x_0 := k^{-1}$, $y_1 := \sqrt{x_0}$ and y_0 is undefined. First observation is;

$$y_{n+1} \geq x_{n+1} \iff \frac{y_n\sqrt{x_n} + 1/\sqrt{x_n}}{y_n + 1} \geq \frac{\sqrt{x_n} + 1/\sqrt{x_n}}{2} \iff x_n \geq 1 \tag{4.8}$$

where we used $x_n, y_n \geq 1$. Also;

$$\sqrt{x_n} \geq y_{n+1} \iff \sqrt{x_n} - \frac{1}{\sqrt{x_n}} \geq 0 \tag{4.9}$$

Collect together as;

$$1 \leq x_{n+1} \leq y_{n+1} \leq \sqrt{x_n} \leq x_n \tag{4.10}$$

Moreover;

$$x_{n+1} - 1 = \frac{(x_n - 1)^2}{2\sqrt{x_n}(1 + \sqrt{x_n})^2} \leq \frac{1}{8}(x_n - 1)^2 \tag{4.11}$$

and

$$\begin{aligned} y_{n+1} - 1 &= \frac{(y_n - 1)(x_n - 1)}{(y_n + 1)(\sqrt{x_n} + 1)} + \frac{2(x_{n+1} - 1)}{y_n + 1} \\ &\leq \frac{1}{4}(y_n - 1)^2 + \frac{1}{8}(x_n - 1)^2 \leq \frac{3}{8}(y_n - 1)^2 \end{aligned} \tag{4.12}$$

This establishes the quadratic convergence of x_n and y_n to 1. Now it is apparent that;

$$\pi = 2\sqrt{2}\frac{M^3(1, 1/\sqrt{2})}{\dot{M}(1, 1/\sqrt{2})} \tag{4.13}$$

and since both M and $\dot{M}$ are quadratically computable, so is π.

In the next section, we turn this identity into an explicit algorithm but before lets turn our attention back to $\dot{a}_n$, and $\dot{b}_n$. We left to show $\dot{b}_n$ decreases.

$$\dot{b}_{n+1} \leq \dot{b}_n \iff \dot{a}_n\sqrt{\frac{a_n}{b_n}} + \dot{b}_n\sqrt{\frac{b_n}{a_n}} \leq 2\dot{b}_n \tag{4.14}$$

or in Legendre Form;

$$\dot{b}_{n+1} \leq \dot{b}_n \iff \frac{1}{y_n}\frac{1}{\sqrt{x_n}} + \sqrt{x_n} \leq 2 \impliedby \frac{1}{x_n} + \sqrt{x_n} \leq 2 \tag{4.15}$$

We know that $x_n \to 1$ therefore write $x_n = 1 + \epsilon_n$ where $\epsilon_n \to 0$. Then;

$$\frac{1}{1 + \epsilon_n} + \sqrt{1 + \epsilon_n} = 2 - \frac{\epsilon_n}{2} + O(\epsilon_n^2) \leq 2 \tag{4.16}$$

Now it is apparent that $\dot{b}_n$ decreases, at least eventually.

5. Algorithm for π

We will use derived AGM and provide estimates on convergence.

Algorithm 1. Let $x_0 := \sqrt{2}$, $\pi_0 := 2 + \sqrt{2}$ and $y_1 := 2^{1/4}$. Define

(i) $x_{n+1} := \dfrac{1}{2}\left(\sqrt{x_n} + \dfrac{1}{\sqrt{x_n}}\right) \qquad n \geq 0$

(ii) $y_{n+1} := \dfrac{1}{2}\dfrac{y_n\sqrt{x_n} + 1/\sqrt{x_n}}{y_n + 1} \qquad n \geq 1$

(iii) $\pi_n := \pi_{n-1}\dfrac{x_n + 1}{y_n + 1} \qquad n \geq 1$

Then π_n decreases monotonically to π. Moreover, for $n \geq 0$;

$$\frac{3}{2}(y_{n+1} - x_{n+1}) \leq \pi_n - \pi \leq \frac{7}{4}(y_{n+1} - x_{n+1}) \tag{5.4}$$

$$\pi_{n+1} - \pi \leq \frac{1}{10}(\pi_n - \pi)^2 \tag{5.5}$$

and for $n \geq 2$,

$$\pi_n - \pi \leq 10^{-2^{n+1}} \tag{5.6}$$

Proof. Define π_n as;

$$\pi_n := 2\sqrt{2}\frac{b_{n+1}^2 a_{n+1}}{\dot{a}_{n+1}} \tag{5.7}$$

where $a_0 := 1$, $b_0 := 1/\sqrt{2}$ and hence $\pi_0 := 2 + \sqrt{2}$. Then by (4.13) $\pi_n \to \pi$. Notice;

$$\frac{\pi_n}{\pi_{n-1}} = \frac{(b_{n+1}/b_n)^2(a_{n+1}/a_n)}{\dot{a}_{n+1}/\dot{a}_n} = \frac{1 + x_n}{1 + y_n} \tag{5.8}$$

Since $y_n \geq x_n \geq 1$, it is obvious that π_n decreases. Now observe;

$$y_{n+1} - x_{n+1} = \frac{(y_n - 1)(x_n - 1)}{2\sqrt{x_n}(1 + y_n)} \leq \frac{1}{8}(y_n - x_n)^2 \tag{5.9}$$

For convenience define, $\tilde{y} := y_n - 1, \tilde{x} := x_n - 1$;

$$\begin{aligned} \frac{\tilde{y}\tilde{x}}{2\sqrt{1+\tilde{x}}(2+\tilde{y})} \leq \frac{1}{8}(\tilde{y} - \tilde{x})^2 \impliedby\ & 2\tilde{y}\tilde{x} \leq (\tilde{y} - \tilde{x})^2 \\ \iff\ & 0 \leq \tilde{y}^2 - 4\tilde{y}\tilde{x} + \tilde{x}^2 \\ \iff\ & 0 \leq 1 - 4\frac{\tilde{x}}{\tilde{y}} + \frac{\tilde{x}^2}{\tilde{y}^2} \end{aligned} \tag{5.10}$$

hence (5.9) is true provided;

$$\frac{x_n - 1}{y_n - 1} < 2 - \sqrt{3} \tag{5.11}$$

It is also necessary to show that (5.11) is true but first note that;

$$\begin{aligned} 1 + y_n x_n - \sqrt{x_n}(1 + y_n) &\geq (\sqrt{x_n} - 1)(y_n - 1) \\ \iff\ y_n x_n - 2y_n\sqrt{x_n} + y_n \geq 0 &\iff (\sqrt{x_n} - 1)^2 \geq 0 \end{aligned} \tag{5.12}$$

Back to (5.11);

$$\begin{aligned} \frac{x_{n+1} - 1}{y_{n+1} - 1}\left(\frac{\sqrt{x_n}}{\sqrt{x_n}}\right) &= \left(\frac{1}{2}\right)\frac{(\sqrt{x_n} - 1)^2(1 + y_n)}{1 + y_n x_n - \sqrt{x_n}(1 + y_n)} \\ &\leq \left(\frac{1}{2}\right)\frac{(\sqrt{x_n} - 1)^2(1 + y_n)(\sqrt{x_n} + 1)}{(\sqrt{x_n} - 1)(y_n - 1)(\sqrt{x_n} + 1)} \\ &\leq \left(\frac{1 + y_n}{4}\right)\frac{x_n - 1}{y_n - 1} \end{aligned} \tag{5.13}$$

On noting that $(x_1 - 1)/(y_1 - 1) \simeq 0.08 \leq 0.26$, (5.11) is always true.

Next;

$$\pi_n - \pi_{n+1} = \pi_n\frac{y_{n+1} - x_{n+1}}{y_{n+1} + 1} \leq \frac{\pi_n}{2}(y_{n+1} - x_{n+1}) \tag{5.14}$$

Therefore;

$$\begin{aligned} \pi_n - \pi = \sum_{k=0}^{\infty}\pi_{n+k} - \pi_{n+k+1} &\leq \frac{1}{2}\sum_{k=0}^{\infty}\pi_{n+k}(y_{n+k+1} - x_{n+k+1}) \\ &\leq \frac{\pi_n}{2}\sum_{k=1}^{\infty}(y_{n+k} - x_{n+k}) \end{aligned} \tag{5.15}$$

To make steps clear, continue from last expression;

$$
\begin{aligned}
\frac{\pi_n}{2}\sum_{k=1}^{\infty}(y_{n+k}-x_{n+k}) &\le \frac{\pi_n}{2}\sum_{k=1}^{\infty}\left[\prod_{i=0}^{k-2}\frac{1}{8^{2^i}}\right](y_{n+1}-x_{n+1})^{2^{k-1}} \\
&\le \frac{\pi_n}{2}\left\{(y_{n+1}-x_{n+1})+\sum_{k=2}^{\infty}\frac{1}{8^{2^{k-2}}}(y_{n+1}-x_{n+1})^{2^{k-1}}\right\} \\
&\le \frac{\pi_n}{2}\left\{(y_{n+1}-x_{n+1})+\sum_{k=2}^{\infty}\frac{1}{8^{k-1}}(y_{n+1}-x_{n+1})^{k}\right\} \\
&= \frac{\pi_n}{2}(y_{n+1}-x_{n+1})\sum_{k=0}^{\infty}\frac{1}{8^{k}}(y_{n+1}-x_{n+1})^{k}
\end{aligned}
\tag{5.16}
$$

As a result we get;

$$
\pi_n - \pi \le \frac{\pi_n}{2}\frac{y_{n+1}-x_{n+1}}{1-\frac{1}{8}(y_{n+1}-x_{n+1})} \tag{5.17}
$$

From (5.14) and since π_n monotone decreases;

$$
\pi_n - \pi \ge \pi_n - \pi_{n+1} \ge \pi\frac{y_{n+1}-x_{n+1}}{1+y_{n+1}} \tag{5.18}
$$

This gives the result;

$$
\frac{3}{2}(y_{n+1}-x_{n+1}) \le \pi_n - \pi \le \frac{7}{4}(y_{n+1}-x_{n+1}) \tag{5.19}
$$

where we note $\pi_0/2(1-\frac{1}{8}(y_1-x_1)) \simeq 1.745 < 1.75$. With (5.9);

$$
\pi_{n+1} - \pi \le \frac{7}{4}(y_{n+2}-x_{n+2}) \le \frac{7}{32}(y_{n+1}-x_{n+1})^2 \le \frac{1}{10}(\pi_n-\pi)^2 \tag{5.20}
$$

Lastly;

$$
\pi_n - \pi \le 10^{-\left(\sum_{i=0}^{k-1}2^i\right)}(\pi_{n-k}-\pi)^{2^k} = 10^{-2^k+1}(\pi_{n-k}-\pi)^{2^k} \le 10^{-2^{k+3}} \tag{5.21}
$$

Last inequality is true for $k = n-2$ on checking;

$$
10^8(\pi_2-\pi) \simeq 0.737 < 1 \tag{5.22}
$$

Notice for this calculation we start with $k \ge 0$ hence (5.6) follows for $n \ge 2$. □

The first eleven iterations give $1,\ 3,\ 8,\ 19,\ 40,\ 83,\ 170,\ 345,\ 694,\ 1393,\ 2789$ digits of π.

Because it is expected to be seen, here you have first 4920 digits of π;

3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679821480865132823066
4709384460955058223172535940812848111745028410270193852110555964462294895493038196442881097566593344612847564823378678316527120190914564856692346034861045432664821339360726024914127372458700660631558817488152092096282925409171536436789259036001133053054882046652138414695194151160943305727036575959195309218611738193261179310511854807446237996274956735188575272489122793818301194912983367336244065664308602139494639522473719070217986094370277053921717629317675238467481846766940513200056812714526356082778577134275778960917363717872146844090122495343014654958537105079227968925892354201995611212902196086403441815981362977477130996051870721134999999837297804995105973173281609631859502445945534690830264252230825334468503526193118817101000313783875288658753320838142061717766914730359825349042875546873115956286388235378759375195778185778053217122680661300192787661119590921642019893809525720106548586327886593615338182796823030195203530185296899577362259941389124972177528347913151557485724245415069595082953311686172785588907509838175463746493931925506040092770167113900984882401285836160356370766010471018194295559619894676783744944825537977472684710404753464620804668425906949129331367702898915210475216205696602405803815019351125338243003558764024749647326391419927260426992279678235478163600934172164121992458631503028618297455570674983850549458858692699569092721079750930295532116534498720275596023648066549911988183479775356636980742654252786255181841757467289097777279380008164706001614524919217321721477235014144197356854816136115735255213347574184946843852332390739414333454776241686251898356948556209921922218427255025425688767179049460165346680498862723279178608578438382796797668145410095388378636095068006422512520511739298489608412848862694560424196528502221066118630674427862203919494504712371378696095636437191728746776465757396241389086583264599581339047802759009946576407895126946839835259570982582262052248940772671947826848260147699090264013639443745530506820349625245174939965143142980919065925093722169646151570985838741059788595977297549893016175392846813826868386894277415599185592524595395943104997252468084598727364469584865383673622262609912460805124388439045124413654976278079771569143599770012961608944169486855584840635342207222582848864815845602850601684273945226746767889525213852254995466672782398645659611635488623057745649803559363456817432411251507606947945109659609402522887971089314566913686722874894056010150330861792868092087476091782493858900971490967598526136554978189312978482168299894872265880485756401427047755513237964145152374623436454285844479526586782105114135473573952311342716610213596953623144295248493718711014576540359027993440374200731057853906219838744780847848968332144571386875194350643021845319104848100537061468067491927819119793995206141966342875444064374512371819217999839101591956181467514269123974894090718649423196156794520809514655022523160388193014209376213785595663893778708303906979207734672218256259966150142150306803844773454920260541466592520149744285073251866600213243408819071048633173464965145390579626856100550810665879699816357473638405257145910289706414011097120628043903975951567715770042033786993600723055876317635942187312514712053292819182618612586732157919841484882916447060957527069572209175671167229109816909152801735067127485832228718352093539657251210835791513698820914442100675103346711031412671113699086585163983150197016515116851714376576183515565088490998985998238734552833163550764791853589322618548963213293308985706420467525907091548141654985946163718027098199430992448895757128289059232332609729971208443357326548938239119325974636673058360414281388303203824903758985243744170291327656180937734440307074692112019130203303801976211011004492932151608424448596376698389522868478312355265821314495768572624334418930396864262434107732269780280731891544110104468232527162010526522721116603966655730925471105578537634668206531098965269186205647693125705863566201855810072936065987648611791045334885034611365768675324944166803962657978771855608455296541266540853061434443185867697514566140680070023787765913440171274947042056223053899456131407112700040785473326993908145466464588079727082668306343285878569830523580893306575740679545716377525420211495576158140025012622859413021647155097925923099079654737612551765675135751782966645477917450112996148903046399471329621073404375189573596145890193897131117904297828564750320319869151402870808599048010941214722131794764777262241425485454033215718530614228813758504306332175182979866223717215916077166925474873898665494945011465406284336639379003976926567214638530673609657120918076383271664162748888007869256029022847210403172118608204190004229661711963779213375751149595015660496318629472654736425230817703675159067350235072835405670403867435136222247715891504953.1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679821480865132823066

from π_{11} of Algorithm 1.

Appendix A. Proofs

Some of the proofs are given as appendix.

Lemma. *$2^{-n}b_n^{-2}\boldsymbol{d}\log(a_n/c_n)$ is independent of n.*

Proof. First observe that;

$$\mathbf{d}\log\left(\frac{a+b}{a-b}\right) = 2\frac{a\mathbf{d}b - b\mathbf{d}a}{a^2-b^2} = \frac{2a^2}{a^2-b^2}\mathbf{d}\left(\frac{b}{a}\right) \tag{A.1}$$

Call expression in lemma as $s(n)$ and because c_n is defined from a_{n-1} and b_{n-1} express $s(n)$ as;

$$s(n) = 2^{-n}(a_{n-1}b_{n-1})^{-1}\mathbf{d}\log\left(\frac{a_{n-1}+b_{n-1}}{a_{n-1}-b_{n-1}}\right) \tag{A.2}$$

We want to show $s(n) = s(n+1)$;

$$s(n+1) = 2^{-n}(a_{n-1}+b_{n-1})^{-1}(a_{n-1}b_{n-1})^{-1/2}\mathbf{d}\log\left(\frac{\sqrt{a_{n-1}}+\sqrt{b_{n-1}}}{\sqrt{a_{n-1}}-\sqrt{b_{n-1}}}\right)^2 \tag{A.3}$$

Apply (A.1) to both $s(n)$ and $s(n+1)$ to get;

$$\begin{aligned} s(n) &= 2^{-n}(a_{n-1}b_{n-1})^{-1}\frac{2a_{n-1}^2}{a_{n-1}^2-b_{n-1}^2}\mathbf{d}\left(\frac{b_{n-1}}{a_{n-1}}\right) \\ s(n+1) &= 2^{-n}(a_{n-1}+b_{n-1})^{-1}(a_{n-1}b_{n-1})^{-1/2}\frac{a_{n-1}}{a_{n-1}-b_{n-1}}\mathbf{d}\left(\sqrt{\frac{b_{n-1}}{a_{n-1}}}\right) \end{aligned} \tag{A.4}$$

It is left to show equality algebraicly. □

Definition 5.

$$\begin{aligned} \Gamma(x) &:= \int_0^\infty e^{-t}t^{x-1}dt & \mathrm{re}(x) > 0 \\ \beta(x,y) &:= \int_0^1 t^{x-1}(1-t)^{y-1}dt & \mathrm{re}(x), \mathrm{re}(y) > 0 \end{aligned} \tag{A.5}$$

By above definitions, Γ satisfies the functional relation

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{A.6}$$

and relationship between Γ and β is

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{A.7}$$

Theorem.

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}} \quad \textit{and} \quad E\left(\frac{1}{\sqrt{2}}\right) = \frac{4\Gamma^2(\frac{3}{4})+\Gamma^2(\frac{1}{4})}{8\sqrt{\pi}} \tag{A.8}$$

Proof. From definition of $K(k)$(1.16);

$$K\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(2-t^2)}} \tag{A.9}$$

The change of variables;

$$x^2 := t^2/(2-t^2) \quad \text{and} \quad dx = \frac{2dt}{(2-t^2)^{3/2}} \quad \text{and} \quad 1-x^4 = 4\frac{(1-t^2)}{(2-t^2)^2} \tag{A.10}$$

Organize terms as;

$$\sqrt{2}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(2-t^2)}} = \sqrt{2}\int_0^1 \frac{2dt}{\sqrt{4(1-t^2)/(2-t^2)^2}(2-t^2)^{3/2}} \tag{A.11}$$

then it is immediate to write;

$$K\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}\int_0^1 \frac{dx}{\sqrt{1-x^4}} \tag{A.12}$$

Now simple change of variables with $u := x^4$ results;

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4}\int_0^1 u^{1/4-1}(1-u)^{1/2-1}du = \frac{\sqrt{2}}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right) \tag{A.13}$$

Now start with

$$E\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\int_0^1 \frac{\sqrt{2-t^2}}{\sqrt{1-t^2}}dt \tag{A.14}$$

Then change of variable as;

$$\begin{aligned} x := \sqrt{1-t^2} \quad &\text{and} \quad dx = \frac{-t}{\sqrt{1-t^2}}dt \\ 1-x^4 = t^2(2-t^2) \quad &\text{and} \quad 1+x^2 = 2-t^2 \end{aligned} \tag{A.15}$$

Organize terms as;

$$\frac{1}{\sqrt{2}}\int_0^1 \frac{\sqrt{2-t^2}}{\sqrt{1-t^2}}dt = \frac{1}{\sqrt{2}}\int_1^0 \frac{2-t^2}{\sqrt{t^2(2-t^2)}}\frac{-tdt}{\sqrt{1-t^2}} \tag{A.16}$$

Then it is immediate again to write;

$$E\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx + \frac{1}{\sqrt{2}}\int_0^1 \frac{dx}{\sqrt{1-x^4}} \tag{A.17}$$

Again $u := x^4$ gives;

$$\begin{aligned} E\left(\frac{1}{\sqrt{2}}\right) &= \frac{1}{4\sqrt{2}}\int_0^1 u^{3/4-1}(1-u)^{1/2-1}du + \frac{1}{4\sqrt{2}}\int_0^1 u^{1/4-1}(1-u)^{1/2-1}du \\ &= \frac{1}{4\sqrt{2}}\beta\left(\frac{3}{4},\frac{1}{2}\right) + \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right) \end{aligned} \tag{A.18}$$

Rest of the proof is to use given relations and properties for β and Γ for both K and E. □

Below theorem is from *Principles of Mathematical Analysis, Rudin W.*(Theorem 7.17)

Theorem. *Suppose $\{f_n\}$ is a sequence of functions, differentiable on [a,b] and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f'_n\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and*

$$f'(x) = \lim_{n\to\infty} f'_n(x) \qquad (a \le x \le b) \tag{A.19}$$

Appendix B. Bounded Variation

(Proofs are sketched, not rigirous.)

Definition 6. Let f be real-valued function defined on an interval $[a,b] \in \mathbb{R}$. Total variation of f is defined as;

$$V(f,[a,b]) \equiv \sup_{P\in\mathcal{P}} \left\{ \sum_{i=0}^{n_P-1} |f(x_{i+1}) - f(x_i)| \right\} \tag{B.1}$$

where $\mathcal{P}$ is the set of all partitions of $[a,b]$.

Definition 7. A real-valued function f is called bounded variation(BV function) if total variation is finite. i.e.

$$f \in BV([a,b]) \iff V(f,[a,b]) < \infty \tag{B.2}$$

Theorem 7. *Let $f \in BV(a,c),\ BV(c,b)$. Then $f \in BV(a,b)$ and;*

$$V(f,[a,b]) = V(f,[a,c]) + V(f,[c,b]) \tag{B.3}$$

Proof. (Sketch.) Union of P_1 and P_2 partition of $[a,c]$, $[c,b]$ respectively forms a partition for $[a,b]$. Observe that summation in (B.1) is equal for such partitions. Therefore $V(f,[a,b]) \ge V(f,[a,c]) + V(f,[c,b])$ is trivial. For the reverse inequality, consider any partition P of $[a,b]$. If $c \in P$, it can be decomposed to form P_1 and P_2. If $c \notin P$, then c can be added to partition P to form finer partition and then decomposed to form P_1 and P_2. Hence any partition P(or finer version of P) can be decomposed to two partitions so $V(f,[a,b]) \le V(f,[a,c]) + V(f,[c,b])$. □

Lemma 2. *Let $f \in BV([a,b])$. Then V(f,[a,x]) is an increasing function.*

Proof. Let $x_1 < x_2$. By Theorem (B.3);

$$V(f,[a,x_2]) - V(f,[a,x_1]) = V(f,[x_1,x_2]) \ge 0 \tag{B.4}$$

□

Theorem 8. *$f \in BV([a,b])$ if and only if there exist two bounded monotone increasing function f_1, f_2 such that $f = f_1 - f_2$.*

Proof. Choose $f_1(x) = V(f, [a, x])$. We know that f_1 is increasing. It is left to show $f_2 = f_1 - f$ is also increasing. Let $x_1 < x_2$;

$$f_1(x_2) - f(x_2) - f_1(x_1) + f(x_1) = V(f, [x_1, x_2]) - (f(x_2) - f(x_1)) \geq 0 \tag{B.5}$$

For last inequality, observe that $\{x_1, x_2\}$ is a coarse partition of $[x_1, x_2]$ hence total variation is clearly larger. For monotone increasing, define $f_1' = f_1 + x$ and $f_2' = f_2 + x$.
For the reverse;

$$\begin{aligned} V(f, [a, b]) = V(f_1 - f_2, [a, b]) &\leq V(f_1, [a, b]) + V(f_2, [a, b]) \\ &= f_1(a) - f_1(b) + f_2(a) - f_2(b) < \infty \end{aligned} \tag{B.6}$$

□

Above theorem implies that if $f \in BV([a, b])$, f is Riemann Integrable because monotone functions are Riemann integrable. (See *Darboux-Froda's Theorem* and *Lebesgue Criterion for Riemann Integral* for discussion.)

Appendix C. Fourier Series

For complete discussion, see Apostol, Mathematical Analysis, Second Edition, Chapter 11. *(Proofs are all omitted.)*

Definition 8. Let $S = \{\phi_0, \phi_1, \phi_2, \cdots\}$ be orthonormal on interval I, and assume that $f \in L^2(I)$. The notation

$$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x) \tag{C.1}$$

will mean that the numbers $c_0, c_1, c_2, \cdots$ are given by;

$$c_n = (f, \phi_n) = \int_I f(x) \bar{\phi_n}(x) dx \tag{C.2}$$

The series in (C.1) is called *the Fourier series of f relative to S*, and the numbers $c_0, c_1, c_2, \cdots$ are called *the Fourier coefficients of f relative to S.*

Theorem 9. *(Jordan). If g is of bounded variation on $[0, \delta]$, then*

$$\lim_{\alpha \to \infty} \frac{2}{\pi} \int_0^{\delta} g(t) \frac{\sin(\alpha t)}{t} dt = g(0+) \tag{C.3}$$

Theorem 10. *Assume that $f \in L([0, 2\pi])$ and suppose f has period 2π. Then the Fourier series generated by f will converge for a given value of x if, and only if, for some positive $\delta < \pi$ the following limit exists:*

$$\lim_{n \to \infty} \frac{2}{\pi} \int_0^{\delta} \frac{f(x+t) + f(x-t)}{2} \frac{\sin\left((n + \frac{1}{2})t\right)}{t} \tag{C.4}$$

Assume that $f \in L([0, 1])$ and suppose that f has period 1. Consider a fixed x in $[0, 1]$ and a positive $\delta < 1/2$. Let

$$g(t) = \frac{f(x+t) + f(x-t)}{2} \quad \text{if} \quad t \in [0, \delta] \tag{C.5}$$

and

$$s(x) = g(0+) = \lim_{t \to 0+} \frac{f(x+t) + f(x-t)}{2} \tag{C.6}$$

whenever this limit exists.

Theorem 11. *(Jordan's test) If f is of bounded variation on the compact interval $[x - \delta, x + \delta]$ for some $\delta < \pi$, then the limit $s(x)$ exists and the Fourier series generated by f converges to $s(x)$.*

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