Set Valued HJB Equations

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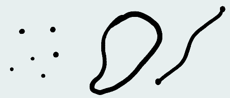
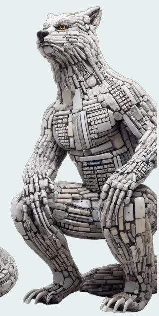
Joint work with Jianfeng Zhang.


$(t, x, \mu, \cdots) \longrightarrow$



Set Valued Frameworks

Geometric Surface Evolutions

Stochastic Viability & Target Problems

Multivariate Dynamic Risk Measures

[8] FEINSTEIN, Z.; RUDLOFF, B. (2015). *Multi-portfolio time consistency for set-valued convex and coherent risk measures*. Finance and Stochastics, **19**, 67-107.

[1] ARARAT, C.; MA, J.; WU, W. (2023). *Set-valued backward stochastic differential equations*. Annals of Applied Probability, **33**(5), 3418-3448.

N-player Games

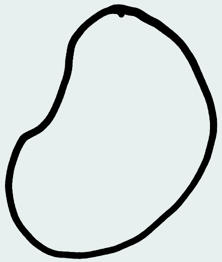
[10] FEINSTEIN, Z.; RUDLOFF, B.; ZHANG, J. (2022). *Dynamic set values for nonzero sum games with multiple equilibria*. Mathematics of Operations Research, **47**, 616-642.

Mean-field Games

[13] İŞERI, M.; ZHANG, J. (2021). *Set values for mean field games*. preprint, arXiv:2107.01661.

Multivariate Control Problems

Set Valued Calculus

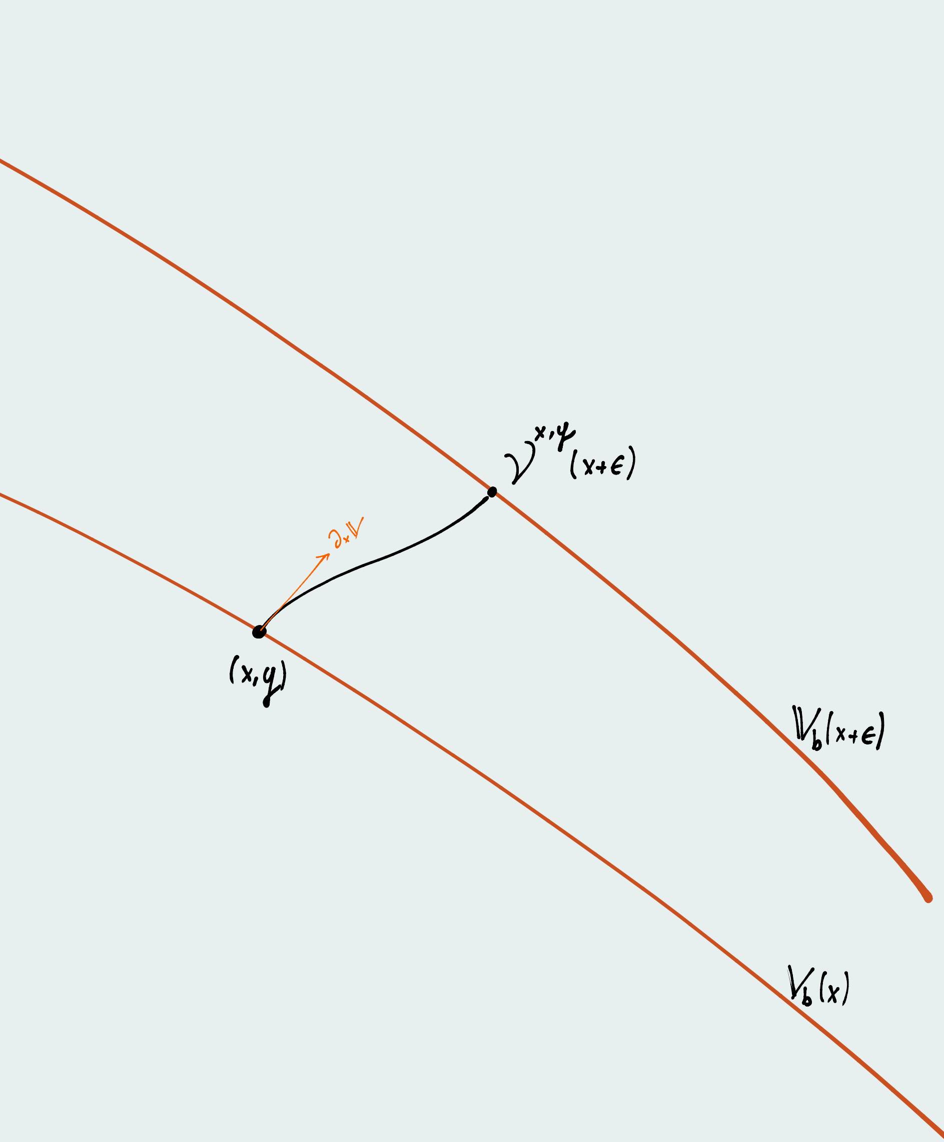
$\mathbb{V}(x) : \mathbb{R}^d \longrightarrow$  $\subset \mathbb{R}^m$

- closed, smooth boundary
- $\mathbb{V}_b(x)$ is the boundary of $\mathbb{V}(x)$

- $\mathcal{G}_{\mathbb{V}} \doteq \{ (x, y) : x \in \mathbb{R}^d , \ y \in \mathbb{V}_b(x) \}$

- $n_{\mathbb{V}}(x,y) \doteq \mathcal{G}_{\mathbb{V}} \to \mathbb{R}^m$ is the outward unit normal vector.

Intrinsic Derivative of $f(x,y): \mathcal{G}_{\mathbb{V}} \to \mathbb{R}$



Defn $\partial_x f(x,y) \doteq \lim_{\epsilon \to 0} \dfrac{f(x+\epsilon, \nu^{x,y}(x+\epsilon)) - f(x,y)}{\epsilon}$

Ex $\partial_x n$ plays a role in Itô's Lemma.

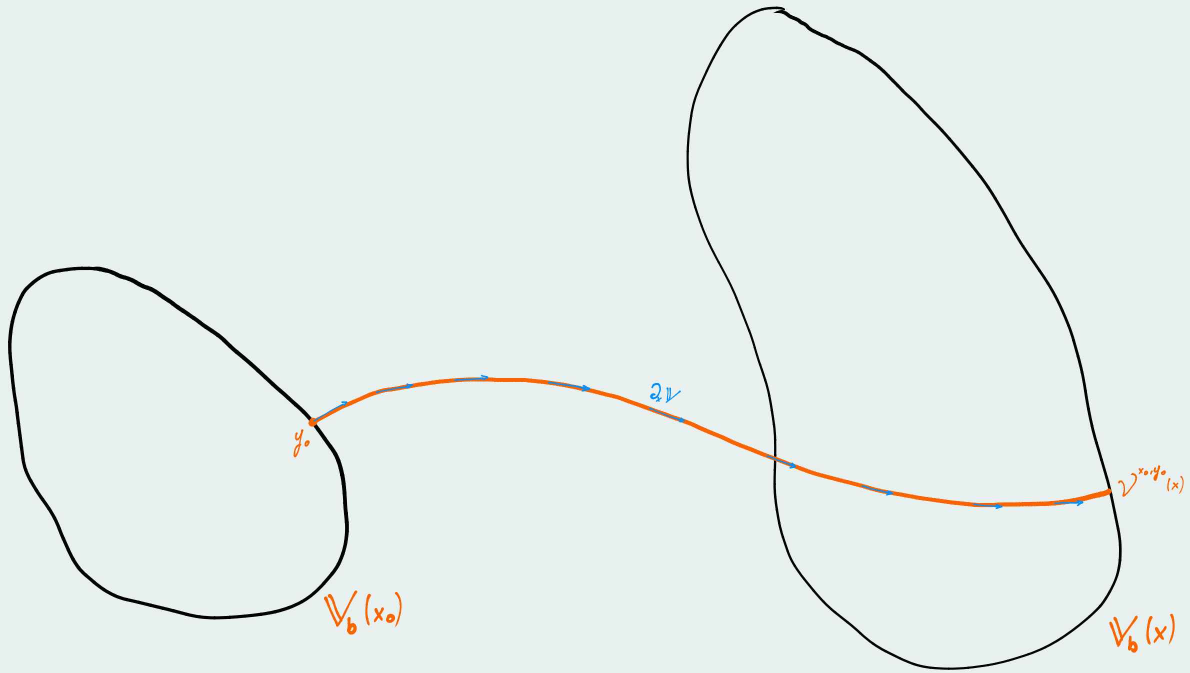
Defn $\partial_x \mathbb{V}(x,y) \doteq \partial_x(y)$ $\ulcorner f(x,y) = y \lrcorner$

Defn $\partial_{xx} \mathbb{V}(x,y) \doteq \partial_x(\partial_x \mathbb{V}(x,y))$

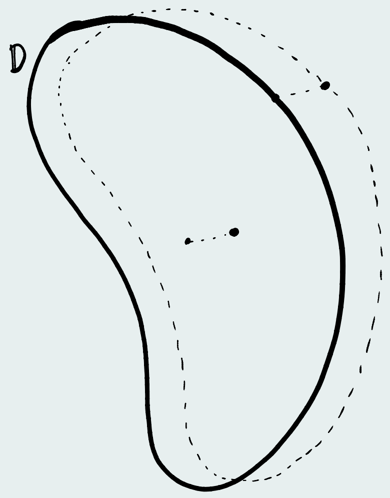
Fundamental Theorem

$$\text{"}\quad \mathbb{V}_b(x) = \mathbb{V}_b(x_0) + \int_x^{x_0} \partial_x \mathbb{V}\big(\tilde{x}, \mathbb{V}_b(\tilde{x})\big)\, d\tilde{x} \quad\text{"}$$

$$\mathbb{V}_b(x) = \left\{ \mathcal{V}^{x_0,y_0}(x) \;:\; \forall y_0 \in \mathbb{V}_b(x_0) \right\} \quad \text{where} \quad \mathcal{V}^{x_0,y_0}(x) = x_0 + \int_{x_0}^{x} \partial_x \mathbb{V}\big(\tilde{x}, \mathcal{V}^{x_0,y_0}(\tilde{x})\big)\, d\tilde{x}$$



Example 1



$$\mathbb{V}(x) = a(x) + \mathbb{D}$$

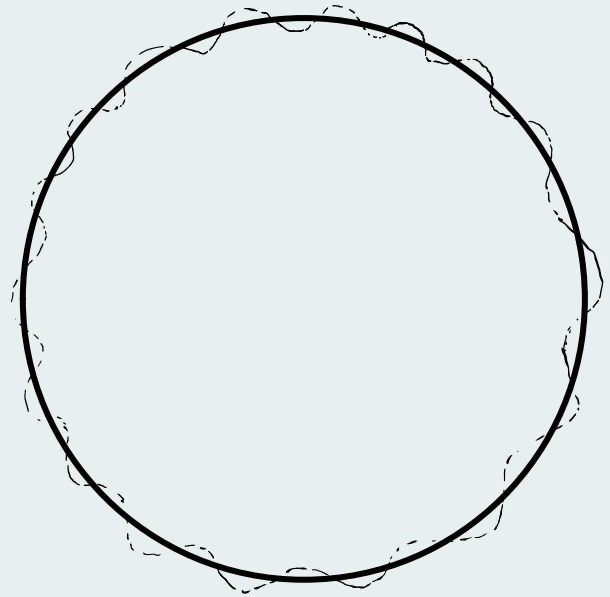
$$\partial_x \mathbb{V} = nn^T \nabla_x a(x)$$

Example 2

$$\mathbb{V}(x) = B(a(x), R(x))$$

$$\partial_x \mathbb{V} = nn^T \nabla_x a + n \nabla_x R$$

Example 3



$$\mathbb{V}_b(x) = \left\{ \left[1 + x\cos\left(1/x + m\theta\right) \right] (\cos\theta, \sin\theta) : \forall \theta \right\}$$

- continuous
- NOT differentiable
- $n_{\mathbb{V}}$ is not continuous.

Surface Evolution Equations

$$\partial_t \mathbb{V}(t,y) = h(t, y, n_{\mathbb{V}}, \partial_y n_{\mathbb{V}})$$

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[7] EVANS, L. C.; SPRUCK, J. (1991). *Motion of level sets by mean curvature. I*. Journal of Differential Geometry, **33**, 635-681.

[4] BARLES, G.; SONER, H. M.; SOUGANIDIS, P. E. (1993). *Front propagation and phase field theory*. SIAM Journal on Control and Optimization, **31**(2), 439-469.

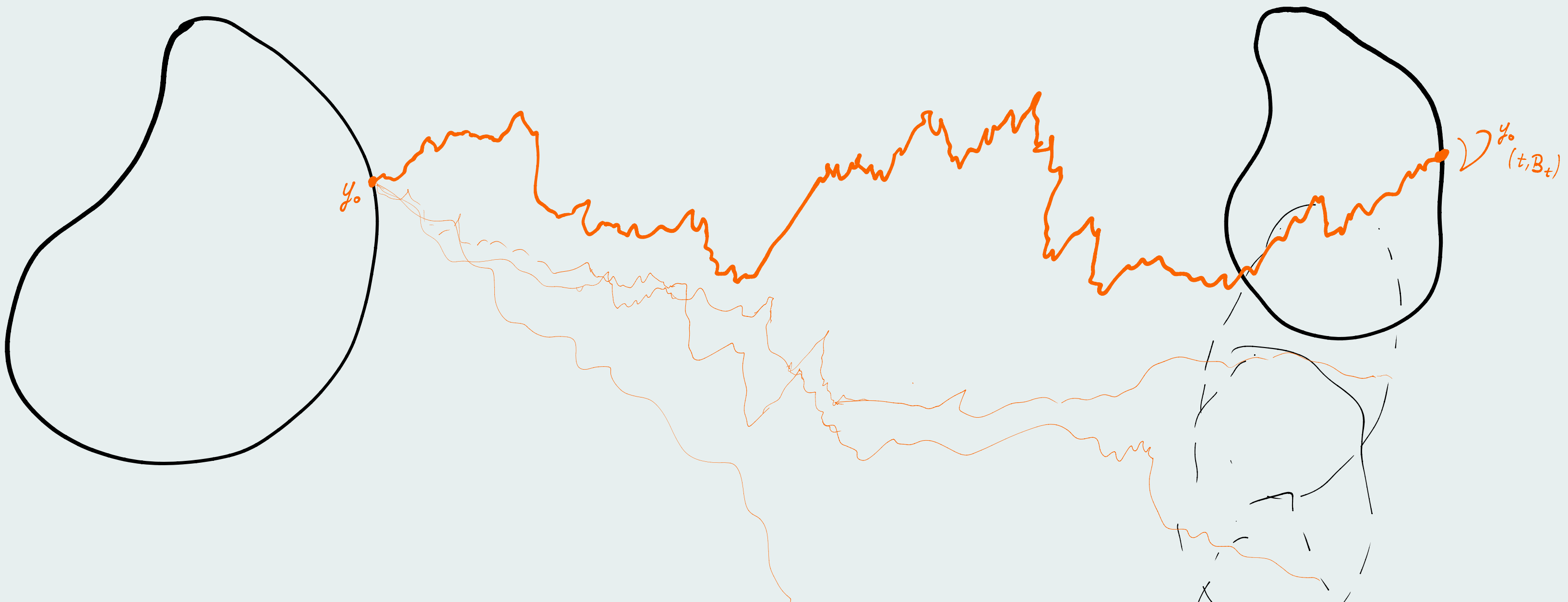
[19] SONER, H. M. (1993). *Motion of a set by the curvature of its boundary*. Journal of Differential Equations, **101**, 313-372.

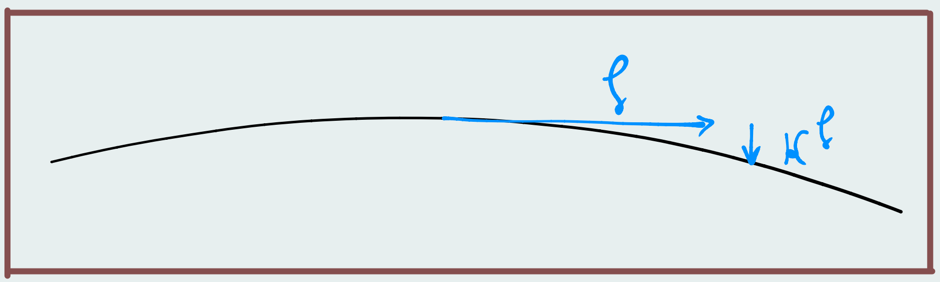
[22] SONER, H. M.; TOUZI, N. (2003). *A stochastic representation for mean curvature type geometric flows*. The Annals of Probability, Vol. 31 No. 3, 1145-1165.

[11] GIGA, Y. (2006). *Surface evolution equations: A level set approach*. Monographs in Mathematics, Birkhäuser Basel.

Itô's Formula

$$\mathbb{V}_b(t,B_t) = \mathbb{V}_0(0,0) + \int_0^t \left[\partial_t \mathbb{V} + \frac{1}{2}\partial_{xx}\mathbb{V} + \kappa^{\ell}\right] ds + \int_0^t \left[\partial_x \mathbb{V} + \ell\right] dB_s$$



Multivariate Control Problem

Dynamics

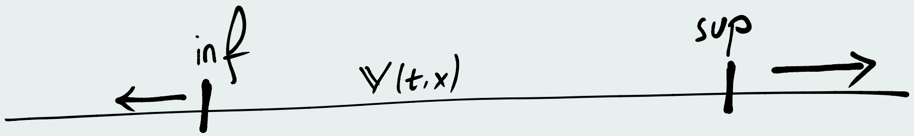
$$X_s^{t,x,\alpha} = x + \int_t^s b(r, X_r^{t,x,\alpha}, \alpha_r)\,dr + \int_t^s \sigma(r, X_r^{t,x,\alpha}, \alpha_r)\,dB_r$$

Value

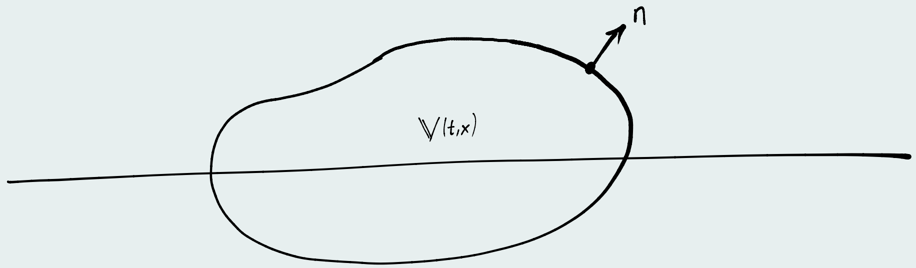
$$Y_s^{t,x,\alpha} = g(X_T^{t,x,\alpha}) + \int_s^T F(r, X_r^{t,x,\alpha}, \alpha_r, Y_r^{t,x,\alpha}, Z_r^{t,x,\alpha})\,dr - \int_s^T Z_r^{t,x,\alpha}\,dB_r$$

Set-Value : $\mathbb{V}(t,x) = \left\{ Y_t^{t,x,\alpha} : \forall \alpha \in \mathcal{A} \right\}$

In one dimension



In higher dimensions



Motivation: Games, Portfolio of assets, Target Problems, Risk of many institutions.

[15] KARNAM, C.; MA, J.; ZHANG, J. (2017). *Dynamic Approaches for Some Time Inconsistent Optimization Problems*. Annals of Applied Probability, **27**, 3435-3477.

Application: Time inconsistent problems

Mean-Variance: $$\sup_\alpha \mathbb{E}\left[X_T^{t,x,\alpha}\right] - \frac{\lambda}{2}\left(\mathbb{E}\left[|X_T^{t,x,\alpha}|^2\right] - \mathbb{E}\left[X_T^{t,x,\alpha}\right]^2\right) = \sup_{y \in \mathbb{V}(t,x)} \varphi(y);$$

$$Y^1 = X_T - \int Z^1 dB, \quad Y^2 = |X_T|^2 - \int Z^2 dB, \quad \varphi(y_1, y_2) \doteq y_1 - \frac{\lambda}{2} y_1 + \frac{\lambda}{2} y_2^2$$

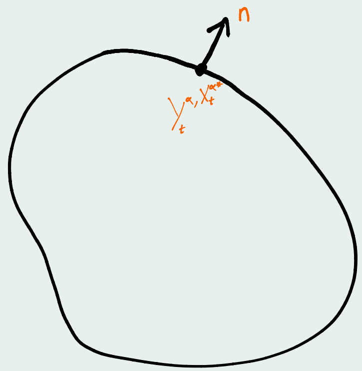
Moving Scalarization $(\varphi(y) \doteq \lambda \cdot y)$:

$$\sup_\alpha Y_0^\alpha \cdot \lambda$$

α^* optimal

$$\sup_\alpha Y_t^{\alpha, X_t^{\alpha^*}} \cdot \lambda$$

$$\sup_\alpha Y_t^{\alpha, X_t^{\alpha^*}} \cdot n\left(t, X_t^{\alpha^*}, Y_t^{\alpha, X_t^{\alpha^*}}\right)$$



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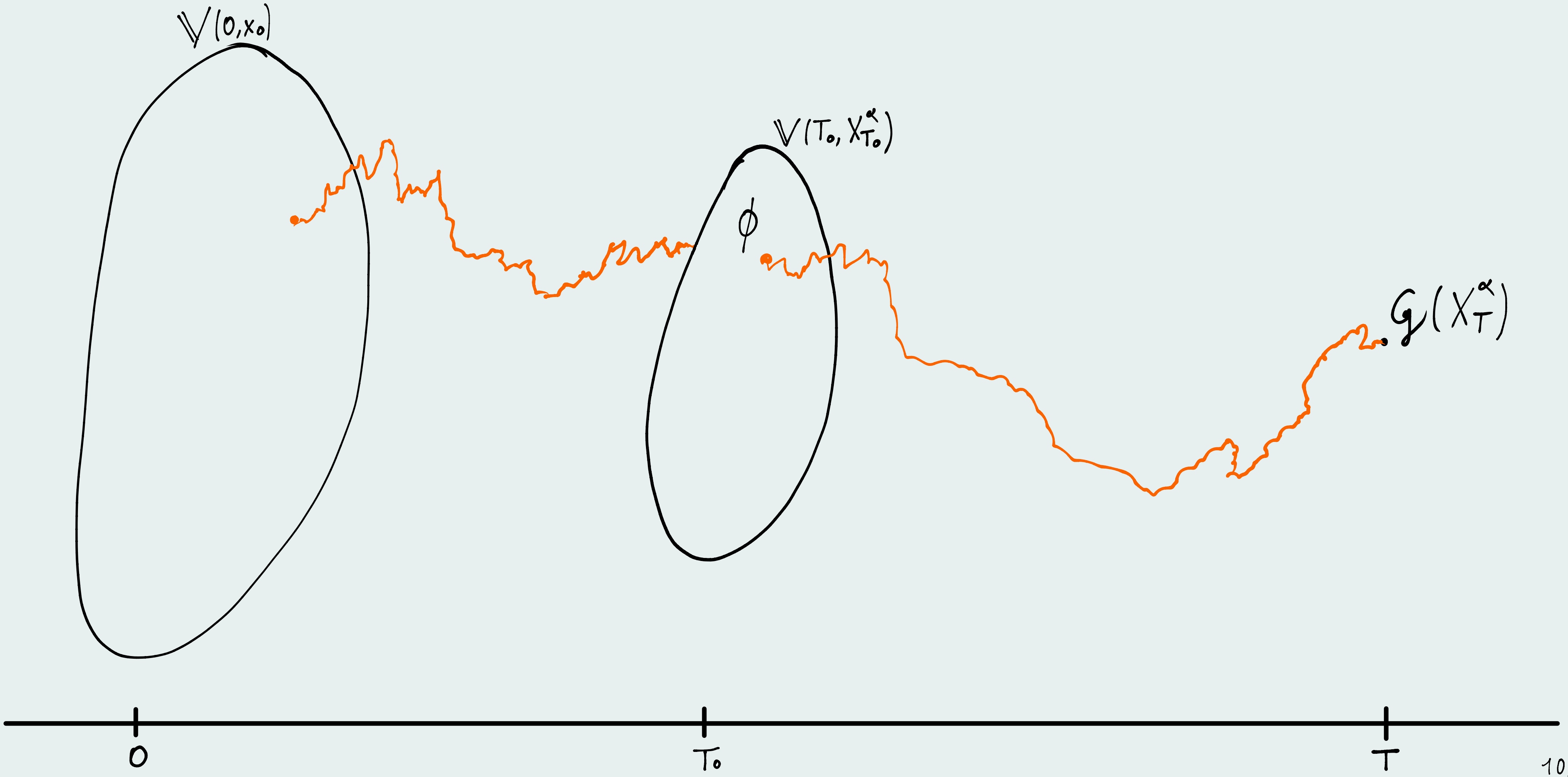
Time - Consistency

$$\mathbb{V}(0,x_0) = \left\{ Y_0^{T_0,\phi,\alpha} \ : \ \forall \alpha \in \mathcal{A} \, , \ \phi \in \mathbb{V}(T_0, X^{\alpha}_{T_0}) \right\}$$



$$\mathcal{HJB} \qquad \sup_{a,\,\xi}\; n\cdot\left[\partial_t \mathbb{V} + \tfrac{1}{2}\partial_{xx}\mathbb{V} + K^{\xi} + F\right](t,x,y,a) = 0 \qquad \forall (t,x,y) \in \mathcal{G}_{\mathbb{V}}$$

$$\mathbb{V}(T,x) = g(x)$$

Main Theorem

(i) Suppose $\mathbb{V} \in C^{1,2}$. Then $\mathbb{V}$ is a classical solution of $\mathcal{HJB}$.

(ii) Suppose $\mathbb{U} \in C^{1,2}$ is a classical solution of $\mathcal{HJB}$. Then $\mathbb{V} = \mathbb{U}$.

(ii') Suppose optimal arguments of Hamiltonian are $\mathcal{I}^*$ and ξ^*.

Then, for any $y \in \mathbb{V}_b(t,x)$, $\exists\, \alpha^*$ s.t. $y = Y_t^{t,x,\alpha^*}$ and $Y_s^{t,x,\alpha^*} \in \mathbb{V}(s, X_s^*)$

Moving Scalarization: Given λ, choose y^λ solving the $\sup_y \lambda \cdot y$.

By verification theorem, construct α^*, X^*, Y^*, and since $Y_t^* \in \mathbb{V}_b(t, X_t^*)$, α^* is optimal for $n(t, X_t^*, Y_t^*)$, $\forall t$.

Mean Variance: Introduce $\psi(y_1, y_2) = (y_1, y_2 - y_1^2)$ and $\tilde{\mathbb{V}}(t,x) = \{\psi(y) : \forall y \in \mathbb{V}(t,x)\}$

$\Longrightarrow \sup_{y \in \mathbb{V}(t,x)} \varphi(y) = \sup_{\tilde{y} \in \tilde{\mathbb{V}}(t,x)} \tilde{y}_1 - \frac{\lambda}{2} \tilde{y}_2$ which is linear! Then, the dynamic mean-variance

$$V_t \doteq \operatorname*{ess\,sup}_\alpha \left\{ \mathbb{E}[X_T^\alpha \mid \mathcal{F}_t] - \frac{\Lambda(t, X^*_{[0,t]})}{2} \mathrm{Var}(X_T^\alpha \mid \mathcal{F}_t) \right\}$$ is time-consistent.

$$\alpha^* = -X_t^* + x_0 + \frac{1}{\lambda} e^T, \quad \Lambda(t, x_{[0,t]}) \doteq \frac{\lambda e^{T-t}}{e^T - \lambda(x_t - x_0)}, \quad V_t = \frac{1}{2}(1 + e^{t-T}) X_t^* + \frac{1}{2}(1 - e^{t-T}) x_0 + \frac{e^T}{2\lambda}(1 - e^{t-T})$$

Thank

You